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MST204 Mathematical Models and Methods.

mathematical models and methods

Unit 31 Fourier Analysis

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Unit 31
Fourier analysis

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Introduction

Joseph Fourier, whose name forms part of the title of this unit, has already been mentioned in this course, in *Unit 12*, on heat transfer. The present unit is about a different part of his work — one which led to results which have become an indispensable part of the theoretical tool kit of applied mathematicians, physicists and engineers.

The fundamental idea is that many functions can be represented by the sum of a series of sinusoidal terms. In other words, sinusoidal functions may be considered to be the building blocks from which other functions may be built up. The point of such a procedure is that in many cases a series of sinusoidal terms is easier to deal with than the original function.

The study of the use of sinusoidal functions to represent other functions is called Fourier analysis and is the subject of this unit.

Study guide

The sections of this unit are intended to be read in the order in which they appear. The first four sections all teach new and assessable material. Section 5 does not teach any new material but consists of problems on the material of the rest of the unit. You should try as many of these as you have time for without looking at the solutions. Then compare your answers with the given solutions, and also look at the solutions of any problems you did not attempt — you may well find them instructive.

There is a television programme associated with this unit. You should preferably have worked through the first three sections before you watch it but, even if you have not done so, it is worth watching the programme rather than missing it deliberately. There is no television section in the main text of this unit but there are notes about the programme in Appendix 3; you should read these notes *before* watching the programme. There is no tape associated with this unit.

1 What Fourier analysis is used for

The Fourier method is to express a function as a sum of sinusoidal terms. This representation is particularly appropriate when the function is *periodic*. In this section we shall look briefly at periodic functions and then, through examples, see why it is useful to be able to express a general periodic function as a sum of sinusoids.

1.1 Periodic functions

A function is said to be periodic if its graph consists of a continually repeated pattern, as in Figure 1. We can express this more formally by saying that a function $f(t)$ is **periodic** when

$$f(t) = f(t + nT) \qquad \text{for all } t, \text{ for } n = \pm 1, \pm 2, \pm 3, \ldots . \tag{1}$$

Definition (1) implies that the graph of $f(t)$ is unaffected by being shifted a distance nT to the right or left.

The positive constant T is called a **period** of the function. Usually T is taken to be the size of the smallest repeating unit in the graph of the function, as in Figure 1, but note that we could take any integer multiple of this basic repeating unit as the period since this, too, will satisfy the definition (1).

When we want to apply the Fourier method to a general periodic function we shall need to be familiar with the periodic behaviour of our building blocks, the sine and cosine functions. For example, the function $\sin(2\pi t/T)$ has period T because

$$\sin\frac{2\pi(t + nT)}{T} = \sin\left(\frac{2\pi t}{T} + 2\pi n\right) = \sin\frac{2\pi t}{T},$$

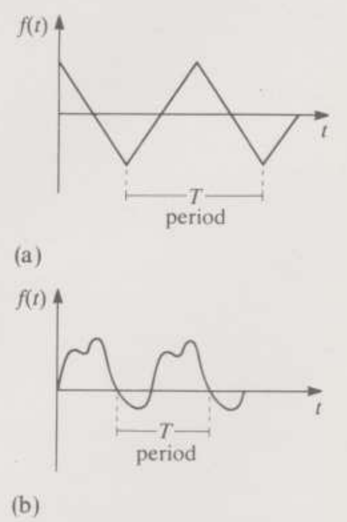


Figure 1. Two periodic functions

so we would expect the function $\sin(2\pi t/T)$ to be useful when we approximate functions with period T.

Question

Does the function $\sin \dfrac{2\pi mt}{T}$, where m is an integer, have T as a period?

Does $\cos \dfrac{2\pi mt}{T}$?

Answer

Yes to both questions, because

$$\sin \frac{2\pi m(t+nT)}{T} = \sin\left(\frac{2\pi mt}{T} + 2\pi mn\right)$$

$$= \sin \frac{2\pi mt}{T}, \quad \text{because } mn \text{ is an integer,}$$

and

$$\cos \frac{2\pi m(t+nT)}{T} = \cos \frac{2\pi mt}{T} \quad \text{by a similar argument.}$$

Strictly speaking these two functions have smallest period T/m but, since any integer multiple of T/m is also a period, the functions also have T as a period.

We shall see later how these functions are used in constructing Fourier representations of functions with period T.

1.2 An application of Fourier analysis

Fourier's idea of using a sum of sinusoidal terms to represent a periodic function turns out to be very useful in many different fields. One of these is the response of mechanical systems to any arbitrary periodic input. Since you have some knowledge of such systems from *Units 7, 8* and *24*, the following examples and exercises are drawn from that field.

Example 1

Assume that the system we want to study can be modelled by the lumped-parameter spring-mass system with linear damping shown in Figure 2. (This is similar to the systems you modelled in *Unit 8*.) The force, F, is periodic but non-sinusoidal. We want to know how the position, x, of the particle, measured from its equilibrium position, will vary with time. How can the Fourier representation of F be used to find x?

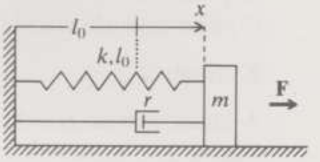


Figure 2. Model system. l_0 is the unstretched length of the spring.

Solution

The equation of motion is

$$m\ddot{x}(t) + r\dot{x}(t) + kx(t) = F(t). \tag{2}$$

Equation (2) can be derived by the methods of *Unit 8*.

It could well be very difficult to solve this equation with F in its original form, as a general periodic function. But now suppose that we can express F as a sum of several sinusoidal terms, say

$$F = F_1 + F_2 + F_3 + \cdots + F_k.$$

We can use the fact that Equation (2) is a *linear* second-order differential equation. This means that the *superposition principle* can be applied. Thus if we can find the steady-state *outputs*, $x_1, x_2, \ldots, x_k$, that would be produced by the component *inputs*, $F_1, F_2, \cdots, F_k$, acting *separately*, then we know that the steady-state output, $x(t)$, produced by their sum, F, will just be the sum of these:

You met the superposition principle in Section 2.4 of *Unit 6*.

$$x(t) = x_1(t) + x_2(t) + \cdots + x_k(t).$$

Now, since the terms $F_1, F_2, \ldots, F_k$ are sinusoidal, we have reduced the problem of finding $x(t)$ to that of finding the output, $x_i(t)$, produced by a single sinusoidal term, $F_i(t)$. What we need to know is the **frequency response** of the system, that is the way in which the output produced by a sinusoidal input of given amplitude depends on the frequency. But we do know how to find the frequency response, using the methods of *Unit 8*.

In this unit I shall use the results we obtained in *Unit 8*, but with some changes in notation. (The notation I use here is common in the sort of engineering applications where Fourier methods are used.) In *Unit 8* we represented a general

sinusoidal input of angular frequency ω by $A_0 \cos(\omega t + \phi_0)$. Here I shall use the input $F_0 \sin \omega t$, which corresponds to $\phi_0 = -\pi/2$. I shall write the corresponding steady-state output as

$$x = A \sin(\omega t + \phi).$$

ϕ is the phase of the output relative to the input, and is often called the **phase shift**.

In *Unit 8* we used the phase lag, $\Delta\phi$, to describe the difference between input and output. The phase shift is $\phi = -\Delta\phi$.

We want to find x for a given input amplitude F_0 and different values of the input angular frequency, ω. We can do this by investigating the behaviour of the *relative response*, A/F_0, and the phase shift, ϕ, for varying values of the angular frequency, ω. This can be represented by the two graphs sketched in Figure 3, which together show the frequency response of the system. The exact shapes of the graphs will depend on the amount of damping. Notice that in this example ϕ is negative for all values of ω.

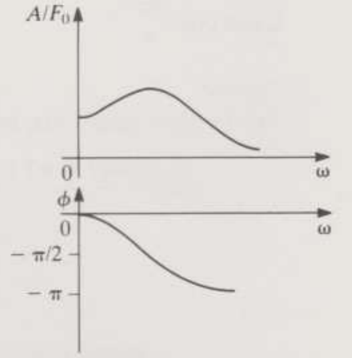


Figure 3. Frequency response for the model system in Figure 2.

Graphs like those in Figure 3 enable us to read off values of A/F_0 and of ϕ for any given angular frequency. So we can find the output, x_i, for each sinusoid, F_i, that makes up the original input. We can then add up all the x_i to find the total response to the periodic input, F.

In the above example we used the fact that the system was modelled by a *linear* second-order differential equation, so that we could use the superposition principle. More complicated systems may lead to higher-order differential equations, but so long as the equations are linear, the superposition principle still applies. We define a **linear system** as a system that can be modelled by a linear differential equation.

The definition of a linear second-order differential equation in *Unit 6* extends naturally to higher orders.

We can now generalize the method of Example 1 to find the response of any *linear* system to a periodic input, F. If we can express F as a sum of sinusoidal terms then all we need is a knowledge of the frequency response of the system. This may be found analytically, as in Example 1, or experimentally, by observing the response of the system to sinusoidal input forces of various frequencies. (Such experiments are standard practice in many practical situations.)

Example 2

A linear system is subject to a periodic input, $f(t)$, which can be expressed as the sum of two sinusoids as follows:

$$f(t) = 2 \sin 5t + 1.5 \sin 10t$$

where t is time.

(i) Figure 4 shows graphs of $y_1 = 2 \sin 5t$ and $y_2 = 1.5 \sin 10t$. Plot $f(t)$ and determine its period.

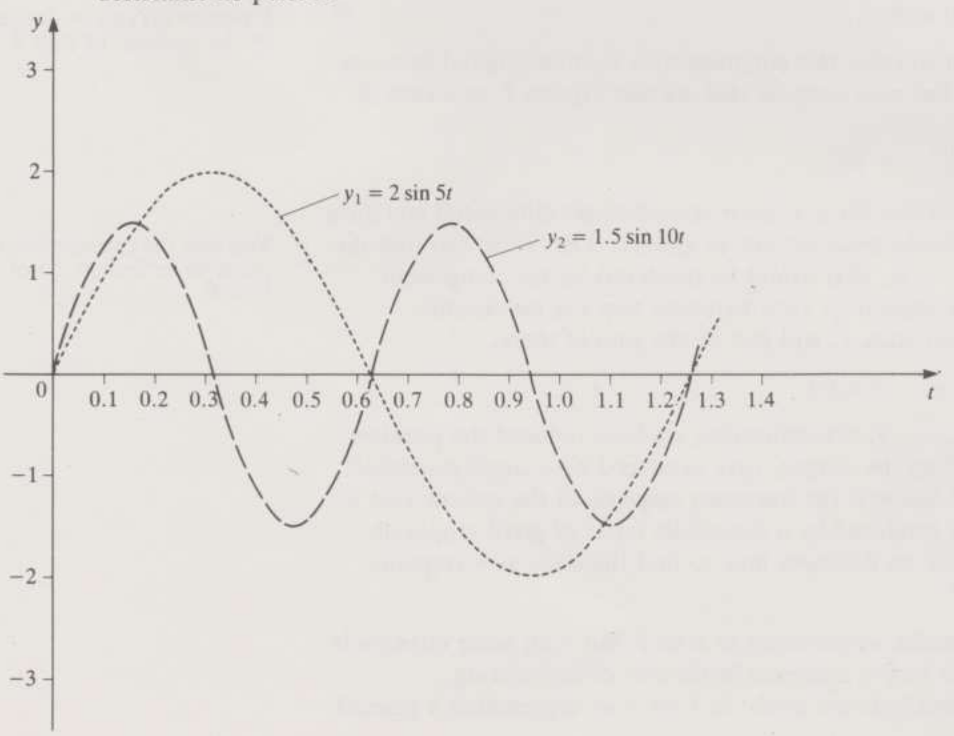


Figure 4

(ii) Figure 5 shows the frequency response of a linear system. Derive an expression for the response of this system to the input $f(t)$.

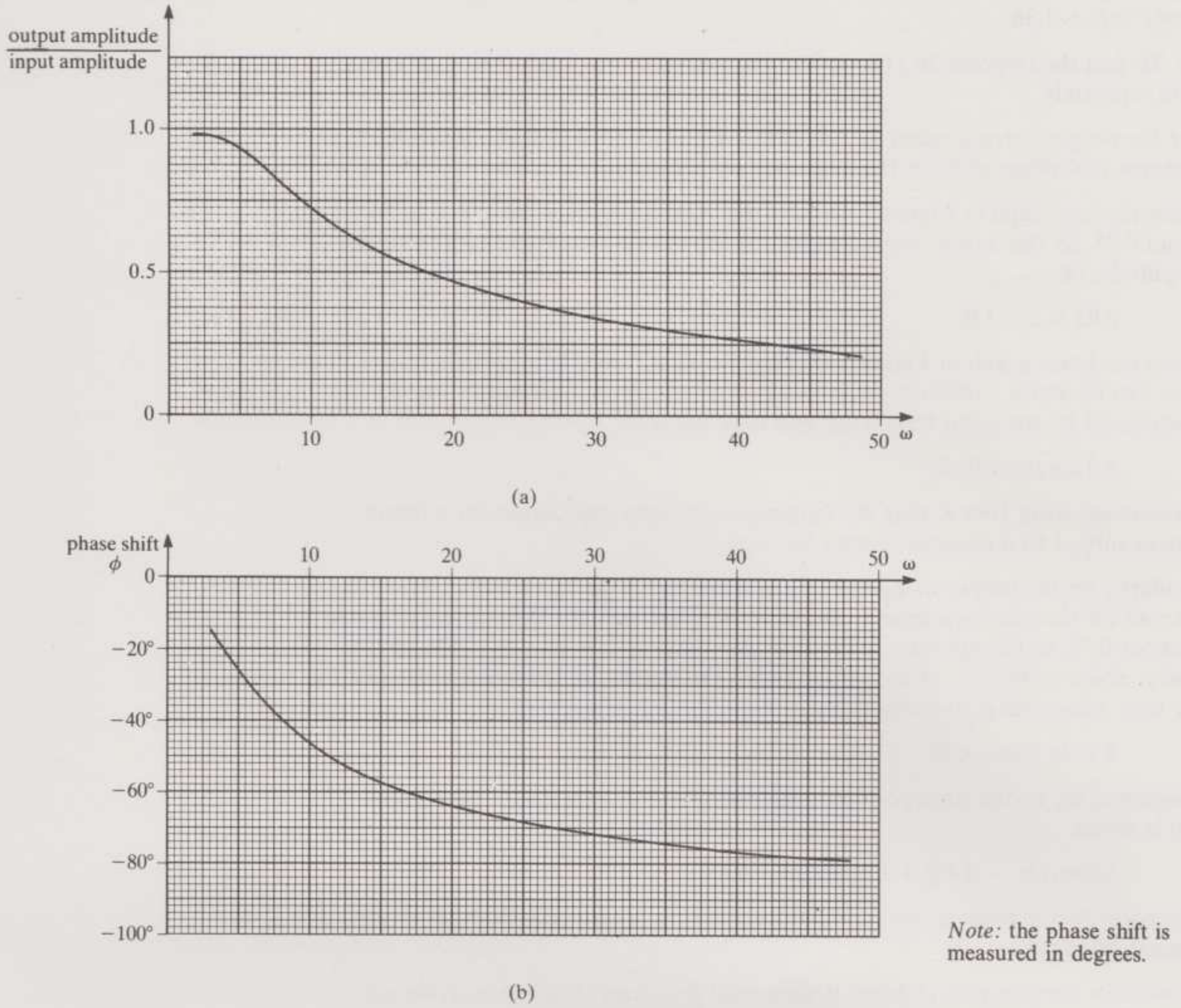


Figure 5. Frequency response of a linear system.

Solution

(i)

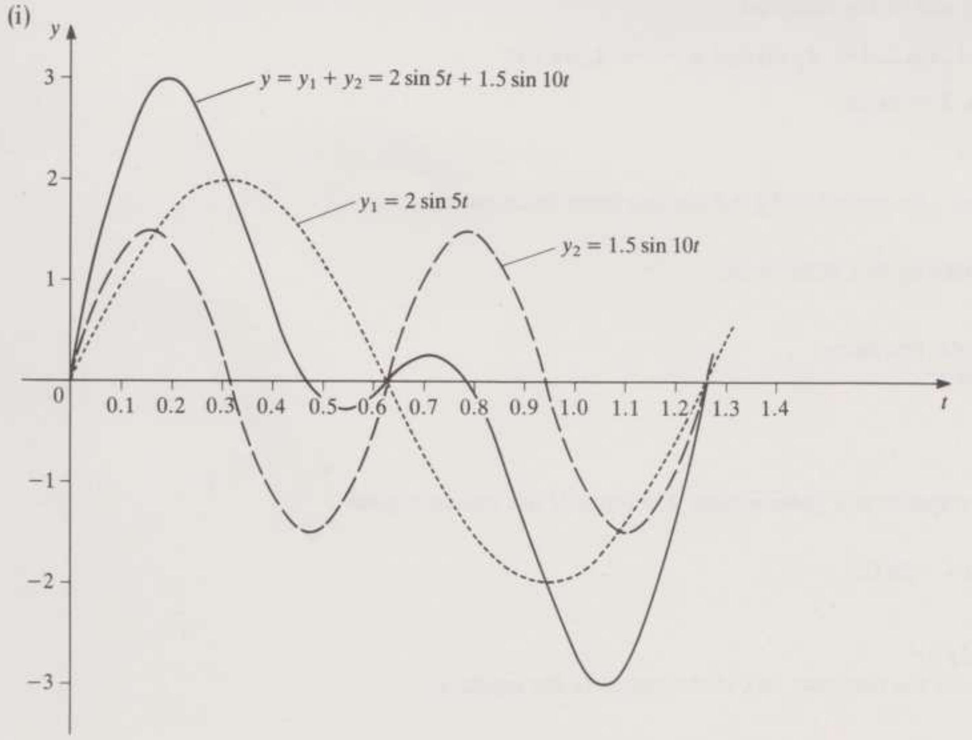


Figure 6

Figure 6 shows both components, y_1 and y_2, of $f(t)$ and their sum, y, which equals $f(t)$. The period, T, of $f(t)$ is equal to that of the component $2\sin 5t$; in other words $2\pi/5 \simeq 1.26$.

(ii) To find the response to $f(t)$ we find the output corresponding to each input term separately.

For the output corresponding to the term $2\sin 5t$ we need to find the relative response and phase shift for the frequency of this input, i.e. $\omega = 5$.

From the top graph in Figure 5 we read the relative response for $\omega = 5$ to be about 0.93. So the output amplitude for the input $2\sin 5t$ is 0.93 times the input amplitude, i.e.

$$0.93 \times 2 \simeq 1.9.$$

From the lower graph in Figure 5 we read the phase shift, ϕ, corresponding to $\omega = 5$ to be about $-26°$, or -0.45 radians; so the component of the output contributed by the input term $2\sin 5t$ is approximately

$$1.9 \sin(5t - 0.45).$$

(Remember, from *Unit 8*, that the frequencies of input and output for a linear system subject to a sinusoidal input are equal.)

Similarly, for the output component corresponding to the input term $1.5\sin 10t$, we read off the relative response and phase shift for $\omega = 10$. The relative response is about 0.72, so the output amplitude at $\omega = 10$ is $0.72 \times 1.5 \simeq 1.1$. The phase shift is about $-46°$, or -0.8 radians. Hence the output component contributed by the term $1.5\sin 10t$ is approximately

$$1.1 \sin(10t - 0.8).$$

Consequently, by the superposition principle, the total response of the system to $f(t)$ is about

$$1.9\sin(5t - 0.45) + 1.1\sin(10t - 0.8).$$

The point that is made in part (i) of the solution is a particular example of a more general principle:

If you take the sum of a number of sinusoidal functions of the form $A_1 \sin \omega t$, $A_2 \sin 2\omega t$, $A_3 \sin 3\omega t$, $\cdots$, $A_k \sin k\omega t$, so that all the frequencies are integral multiples of the lowest frequency, then the sum is periodic and the period is equal to that of the component with the lowest frequency (often called the **fundamental angular frequency**). In other words the function

$$f(t) = A_1 \sin \omega t + A_2 \sin 2\omega t + A_3 \sin 3\omega t + \cdots + A_k \sin k\omega t$$

is periodic and its period is $T = 2\pi/\omega$.

Exercise 1

Suppose the variation with time t (in seconds) of a voltage at a point in an electrical network is given by

$$V(t) = 0.01\sin 2t + 0.005\sin 4t + 0.002\sin 8t.$$

Determine

(i) the fundamental angular frequency;
(ii) the period of the voltage.

[*Solution on p. 23*]

Exercise 2

Figure 7 shows the frequency response of a linear system. The input of this system is given by

$$f(t) = \sin 4t + 2\sin 8t + 3\sin 12t$$

where t is the time in seconds.

(i) What is the period of $f(t)$?
(ii) Derive an expression for the response, $x(t)$, of the system to the input, $f(t)$.

[*Solution on p. 23*]

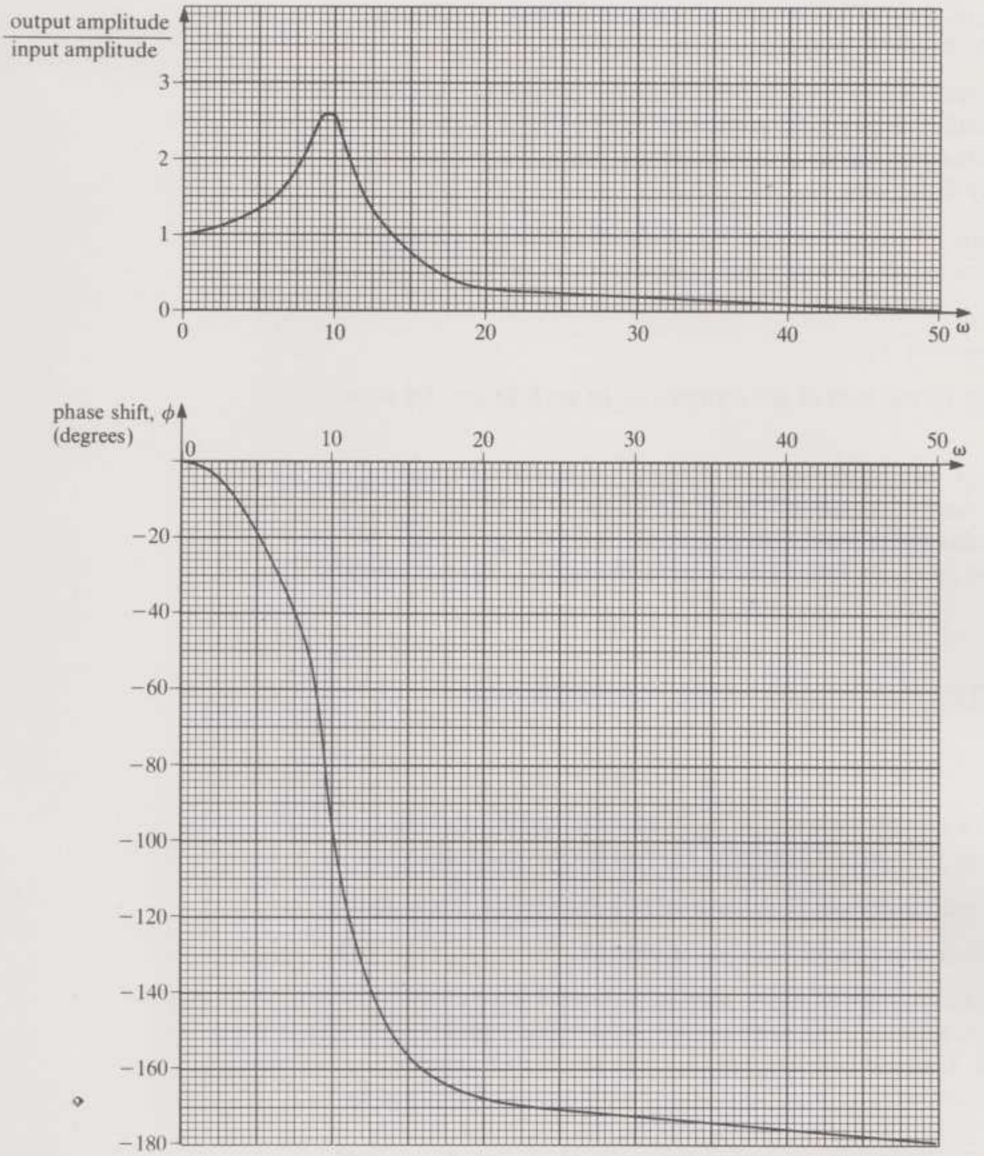


Figure 7

As I have already said, although Figure 2 shows the kind of model that is used in the analysis of mechanical vibrations, and Figures 3 and 7 look like the frequency response associated with such a model, you must not conclude that the procedures in this unit are useful only in the context of mechanical vibrations. On the contrary, they are widely used in such different fields as acoustics, meteorology, electrical network theory, and the design of measuring intruments; and this is by no means a complete list. In addition, Fourier series are mathematically useful in, for example, the solution of differential equations.

See *Unit 32, Partial differential equations.*

So, in order to decide such different matters as the suitability of a given hi-fi amplifier for the reproduction of a particular tape recording or the relationship between the reading produced by a measuring instrument and the actual value of the measured quantity, we need the frequency response of the amplifier or the measuring instrument and a procedure for expressing the signal (i.e. the sound or other quantity of interest) as a sum of sinusoidal functions. The next section is an account of such a procedure.

Summary of Section 1

1. A function f is **periodic** if

$$f(t) = f(t + nT) \qquad \text{for all } t \text{, for } n = \pm 1, \pm 2, \pm 3, \ldots,$$

where T is the **period**.

2. The way that a mechanical system responds to sinusoidal inputs of different frequencies is called the **frequency response** of the system.

3. A **linear system** is one that can be modelled by a linear diffierential equation. A linear second-order differential equation was defined in *Unit 6*—this definition extends naturally to higher-order equations. The superposition principle applies to any linear differential equation.

4. If the frequency response of a linear system is known, the response to any periodic input, F, can be found provided that F can be expressed as a sum of sinusoidal terms,

$$F = F_1 + F_2 + \cdots + F_k.$$

The output, x, will then be the sum of the outputs, x_i, to each sinusoidal input, F_i, taken separately.

5. The sum of a number of sinusoidal terms of different frequencies is a non-sinusoidal function. If the frequencies of the sinusoids are integral multiples of the lowest one, the **fundamental angular frequency**, ω, say, then their sum will be periodic with period $2\pi/\omega$, i.e. the period corresponding to the fundamental angular frequency.

2 Fourier series for periodic functions

2.1 Introduction

The aim of this section is to set up a systematic procedure for writing a given periodic function in terms of the sum of sinusoidal (or 'harmonic') components.

From the last section you will remember that the sum of a number of terms like

$$A_1 \sin \omega t + A_2 \sin 2\omega t + A_3 \sin 3\omega t + \cdots + A_k \sin k\omega t \tag{1}$$

(where $A_1, A_2, A_3, \ldots, A_k$ and ω are constants, with $\omega > 0$ and k a positive integer), will be a periodic function, f, of t, with period $2\pi/\omega$ which equals the smallest period of the term $A_1 \sin \omega t$ with the lowest, or fundamental, angular frequency. What we are aiming at is to *start* with a periodic function f and *end up* with a series like (1), with all the constants taking known values. Before we get down to this, I want to make three adjustments to the series (1), all of them aimed at making it more general, that is to say, capable of representing more functions.

One adjustment arises from the recognition that what we have so far done with terms like $\sin \omega t$ and $\sin 2\omega t$ can also be done with terms like $\cos \omega t$ and $\cos 2\omega t$. In other words, you can get a periodic function by adding up a number of terms like

$$B_1 \cos \omega t + B_2 \cos 2\omega t + B_3 \cos 3\omega t + \cdots + B_k \cos k\omega t. \tag{2}$$

To take account of this fact I shall represent the series which constitutes our initial function by the sum of a number of sine and cosine terms, like this:

$$\begin{aligned} &A_1 \sin \omega t + A_2 \sin 2\omega t + A_3 \sin 3\omega t + \cdots + A_k \sin k\omega t \\ &+ B_1 \cos \omega t + B_2 \cos 2\omega t + B_3 \cos 3\omega t + \cdots + B_k \cos k\omega t. \end{aligned}$$

This can be written more concisely as

$$\sum_{n=1}^{k} (A_n \sin n\omega t + B_n \cos n\omega t).$$

The second change I want to introduce removes another restriction which arises from the fact that the *mean value* of any function like $A \sin n\omega t$ or $B \cos n\omega t$ taken over any interval of length T is equal to zero. Any sum of such terms also has a zero mean value, like the function in Figure 1(a) of Section 1. Now the procedure I am about to show you is not restricted to functions with a zero mean. It can quite easily cope with other mean values. For example, the periodic function sketched in Figure 1(b) of Section 1 has a positive mean value and our procedure is quite capable of dealing with a case like that. It can also cope with a negative

mean value. In order to allow for a non-zero mean, all we need to do is to shift the graph of our function up or down by the required amount, which simply requires an appropriate constant added to the series, so that our series becomes

$$f(t) = M + \sum_{n=1}^{k} (A_n \sin n\omega t + B_n \cos n\omega t) \tag{3}$$

where M is the new constant.

The addition of the term M will not, of course, change the period of the right-hand side of Equation (3). As far as frequency response calculations are concerned (like those we did in Section 1), M is a cosine term with zero frequency and its contribution to the total response is worked out accordingly, as in the example below.

Example 1

Work out the effect on the response of the system described in Exercise 2 of Section 1 of adding a constant term to the input function $f(t)$ so that the new input function becomes

$$f(t)_{\text{new}} = 3 + \sin 4t + 2\sin 8t + 3\sin 12t.$$

Solution

The new constant will add an extra term to the response. It corresponds to a cosine term with zero frequency. From the graph in Figure 7 of Section 1 we see that the amplitude ratio is 1 and the phase shift is zero for $\omega = 0$. So the output term corresponding to 3 in the input is $3 \times \cos(0 + 0) = 3$ and the total response will now be

$$x(t)_{\text{new}} \simeq 3 + 1.2\sin(4t - 0.19) + 4\sin(8t - 0.77) + 4.5\sin(12t - 2.3).$$

There is one more change I need to make to the series (3) so that it can represent a wider class of functions, and that is to extend it to an infinite series, i.e.

$$f(t) = M + \sum_{n=1}^{\infty} (A_n \sin n\omega t + B_n \cos n\omega t). \tag{4}$$

The right-hand side of Equation (4) is known as the **Fourier series** for $f(t)$. Now, we know that this series represents a periodic function with period $2\pi/\omega$. But what we are trying to do is to find this series for *any* given periodic function. Can this be done? And, if it can, how do we find the appropriate values of the constants M, A_n and B_n?

The answer to the first question, for practical purposes, is 'Yes'. (We shall see what conditions the given function must satisfy later, in Subsection 2.3.) But for now we shall assume that our periodic function *can* be expressed as a Fourier series and get on with finding the constants.

2.2 Finding the constants

Let us see how to calculate the constants M, A_1, B_1, A_2, B_2, ... in Equation (4) from our knowledge of the function $f(t)$ itself; this might be given, for example, in the form of a graph. We shall need to use the following integrals.

Some useful integrals

In all cases, m and n stand for non-negative integers and $\omega = 2\pi/T$.

$$\int_{-T/2}^{T/2} \sin n\omega t\, dt = \int_{-T/2}^{T/2} \cos n\omega t\, dt = 0, \qquad n \neq 0. \tag{5a}$$

$$\int_{-T/2}^{T/2} \sin n\omega t \sin m\omega t\, dt = \int_{-T/2}^{T/2} \cos n\omega t \cos m\omega t\, dt = 0, \quad n \neq m. \tag{5b}$$

$$\int_{-T/2}^{T/2} \sin n\omega t \cos m\omega t\, dt = 0. \tag{5c}$$

$$\int_{-T/2}^{T/2} \sin^2 n\omega t\, dt = \int_{-T/2}^{T/2} \cos^2 n\omega t\, dt = \frac{T}{2}, \qquad n \neq 0. \tag{5d}$$

You can check these integrals using the *Handbook*.

Now to find M we integrate every term on both sides of Equation (4) with respect to t between the limits $-T/2$ and $T/2$. Hence

$$\int_{-T/2}^{T/2} f(t)\,dt = \int_{-T/2}^{T/2} M\,dt + \int_{-T/2}^{T/2} A_1 \sin\omega t\,dt + \int_{-T/2}^{T/2} B_1 \cos\omega t\,dt + \int_{-T/2}^{T/2} A_2 \sin 2\omega t\,dt + \int_{-T/2}^{T/2} B_2 \cos 2\omega t\,dt + \cdots.$$

On the right-hand side, every term except the first one is zero according to (5a), so that

$$\int_{-T/2}^{T/2} f(t)\,dt = TM$$

therefore $M = \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} f(t)\,dt.$ (6)

In other words, M is the *mean value* of $f(t)$.

We can find A_1 by multiplying every term on both sides of Equation (4) by $\sin\omega t$ as follows:

$$f(t)\sin\omega t = M\sin\omega t + A_1\sin^2\omega t + B_1\cos\omega t\sin\omega t + A_2\sin 2\omega t\sin\omega t + B_2\cos 2\omega t\sin\omega t + \cdots$$

and now we integrate every one of these terms over t between the limits of $-T/2$ to $T/2$:

$$\int_{-T/2}^{T/2} f(t)\sin\omega t\,dt = \int_{-T/2}^{T/2} M\sin\omega t\,dt + \int_{-T/2}^{T/2} A_1\sin^2\omega t\,dt + \int_{-T/2}^{T/2} B_1\cos\omega t\sin\omega t\,dt + \int_{-T/2}^{T/2} A_2\sin 2\omega t\sin\omega t\,dt + \int_{-T/2}^{T/2} B_2\cos 2\omega t\sin\omega t\,dt + \cdots.$$

If we compare all the terms on the right-hand side with (5a), (5b), (5c) and (5d) we see that they are all zero except the second one. Hence

$$\int_{-T/2}^{T/2} f(t)\sin\omega t\,dt = \int_{-T/2}^{T/2} A_1\sin^2\omega t\,dt = A_1 \times \frac{T}{2}$$

so

$$A_1 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\omega t\,dt.$$

Exercise 1

By multiplying every term on both sides of Equation (4) by $\cos\omega t$ and then integrating each of the resulting terms over time from $-\dfrac{T}{2}$ to $\dfrac{T}{2}$ show that

$$B_1 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\omega t\,dt$$

where T is the period of $f(t)$.

[*Solution on p. 23*]

All the other coefficients are found in the same way: by multiplying all the terms by the sinusoidal function which forms part of the same term as the required coefficient and then integrating over time from $-T/2$ to $T/2$. It follows, for example, that

$$\int_{-T/2}^{T/2} f(t)\sin 2\omega t\,dt = \int_{-T/2}^{T/2} A_2\sin^2 2\omega t\,dt$$

so that

$$A_2 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin 2\omega t\,dt.$$

Similarly

$$B_2 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos 2\omega t\, dt \qquad \text{and so on.}$$

Hence for any value of n

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\, dt \tag{7}$$

and

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\, dt. \tag{8}$$

By following the procedure in the box below, we can now find the Fourier series corresponding to some simple functions.

Procedure 2.2

To find the constants of the Fourier series

$$f(t) = M + \sum_{n=1}^{\infty} (A_n \sin n\omega t + B_n \cos n\omega t) \tag{4}$$

for a periodic function $f(t)$ starting with a graph of $f(t)$ against t.

1. From the graph determine T, the smallest period of the function $f(t)$, and hence the fundamental angular frequency ω by putting $\omega = 2\pi/T$.
2. Write down the equation for $\int_{-T/2}^{T/2} f(t)\, dt$ in terms of t.
3. Find the constant term $M = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, dt$. (6)

   (This is the mean value of $f(t)$ over one complete period.)
4. Find the coefficients $A_1, A_2, A_3, \ldots$ of the sine terms by evaluating
   $$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\, dt \qquad (n = 1, 2, 3, \ldots) \tag{7}$$
   (ω as determined in Step (1) above).
5. Find the coefficients $B_1, B_2, B_3, \ldots$ of the cosine terms by evaluating
   $$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\, dt \qquad (n = 1, 2, 3, \ldots) \tag{8}$$
   (ω as determined in Step (1) above).

Example 2

Derive the Fourier series for the saw-tooth function shown in Figure 1.

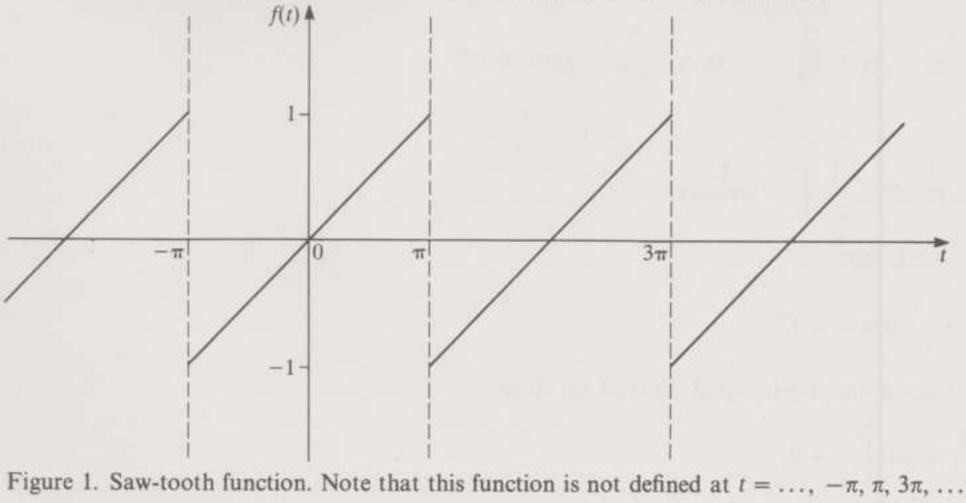


Figure 1. Saw-tooth function. Note that this function is not defined at $t = \ldots, -\pi, \pi, 3\pi, \ldots$.

Solution

First, we find the period, which equals 2π because the graph repeats at intervals of 2π. From this we find the fundamental angular frequency ω to be 1 from the formula $\omega = 2\pi/T$. It follows that the Fourier series has the form

$$f(t) = M + \sum_{n=1}^{\infty} (A_n \sin nt + B_n \cos nt).$$

Now M is the mean value and is given by

$$M = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt \qquad \text{where } \frac{T}{2} = \pi.$$

For t between $-\pi$ and π, $f(t) = t/\pi$ (except for the end-points, $t = -\pi$ and $t = \pi$, which do not affect the value of the integral).

Therefore

$$M = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{t}{\pi}\,dt = \frac{1}{2\pi^2}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi} = 0.$$

(This was to be expected from the symmetry of the graph.)

Again

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{\pi}\sin nt\,dt$$
$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi} t\sin nt\,dt.$$

Now, from the tables of standard integrals in the *Handbook* we find that

$$\int t\sin at\,dt = -\frac{t}{a}\cos at + \frac{1}{a^2}\sin at + C.$$

In our case, $a = n$ and for a definite integral we may omit the constant C, so that

$$A_n = \frac{1}{\pi^2}\left[-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi^2 n}[-\pi\cos n\pi + 0 + (-\pi)\cos(-n\pi) - 0]$$
$$= \frac{-2\pi\cos n\pi}{\pi^2 n} = \frac{-2\cos n\pi}{\pi n}.$$

Now $\cos n\pi = (-1)^n$ so

If you have not met this before, note that
$(-1)^1 = -1$
$(-1)^2 = 1$
and so on.

$$A_n = \frac{-2}{\pi n}(-1)^n = \frac{2}{\pi n}(-1)^{n+1}.$$

(Remember that n is any positive integer.)

So

$$A_1 = \frac{2}{\pi}, \qquad A_2 = \frac{-1}{\pi}, \qquad A_3 = \frac{2}{3\pi}, \qquad A_4 = \frac{-2}{4\pi}, \qquad \text{and so on.}$$

Similarly

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt = \frac{1}{\pi^2}\int_{-\pi}^{\pi} t\cos nt\,dt.$$

Again, we find from the *Handbook* that

$$\int t\cos at\,dt = \frac{t}{a}\sin at + \frac{1}{a^2}\cos at + C.$$

Again, $a = n$ and C is omitted for a definite integral, so that we have

$$B_n = \frac{1}{\pi^2}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_{-\pi}^{\pi} = 0.$$

Hence the required Fourier series consists only of sine terms since $M = B_n = 0$; therefore

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{\pi n}(-1)^{n+1} \sin nt$$
$$= \frac{2}{\pi}\left(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \cdots\right).$$

Notice that the higher the frequency of a term, the lower its amplitude.

Exercise 2

How would the Fourier series derived in Example 2 above be affected if

(i) the height of the 'saw-tooth' were doubled, i.e. the value of $f(t)$ increased from -2 to $+2$ as t increased from $(2m+1)\pi$ to $(2m+3)\pi$ (where m is any integer)?

(ii) the height of the saw-tooth remained the same as in Figure 1 but the period were doubled to 4π?

[*Solution on p. 23*]

2.3 What kind of functions?

What we have done so far is to show that *if* a function $f(t)$ with period T can be expressed in the form

$$f(t) = M + \sum_{n=1}^{\infty} (A_n \sin n\omega t + B_n \cos n\omega t) \tag{4}$$

then the coefficients M, A_n, B_n can be calculated from the formulae (6), (7), (8). To make full use of this result we need to know under what conditions a given periodic function $f(t)$ can be expressed in the above form.

There is a theorem which gives a precise result of this kind; in order to state it we need two new mathematical concepts: *continuous* functions and *piecewise continuous* functions.

A function is said to be **continuous** if its graph is an unbroken curve, that is if there are no gaps in it. For example the sine function is continuous, but the 'saw-tooth' function of Example 2, illustrated in Figure 1, is not, because of the gaps in the graph at $t = \ldots, -\pi, \pi, 3\pi, \ldots$.

A function is said to be **piecewise continuous** if its graph consists of continuous pieces; there must be only a finite number of pieces on any finite part of the domain, and the two pieces ending at each gap must have well-defined end-points even though these points need not themselves be parts of the graph. For example the saw-tooth function illustrated in Figure 1 is piecewise continuous; each piece has well-defined end-points even though these are not part of the graph. On the other hand the function $1/x$ is not piecewise continuous because the pieces do not have well-defined end-points at the gap where $x = 0$.

The graph of $1/x$ is shown in the *Handbook*.

Now we can state the theorem:

Theorem 2.3

If a periodic function f has period T and is piecewise continuous, and if its derivative is also piecewise continuous, then it can be represented by the Fourier series

$$f(t) = M + \sum_{n=1}^{\infty} (A_n \sin n\omega t + B_n \cos n\omega t) \tag{4}$$

where $\omega = 2\pi/T$. The coefficients are given by

$$M = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt \tag{6}$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt \tag{7}$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt. \tag{8}$$

The formula (4) is valid for all values of t at which there is no gap in the graph.

Exercise 3

Derive the Fourier series for the 'square wave' function shown in Figure 2.

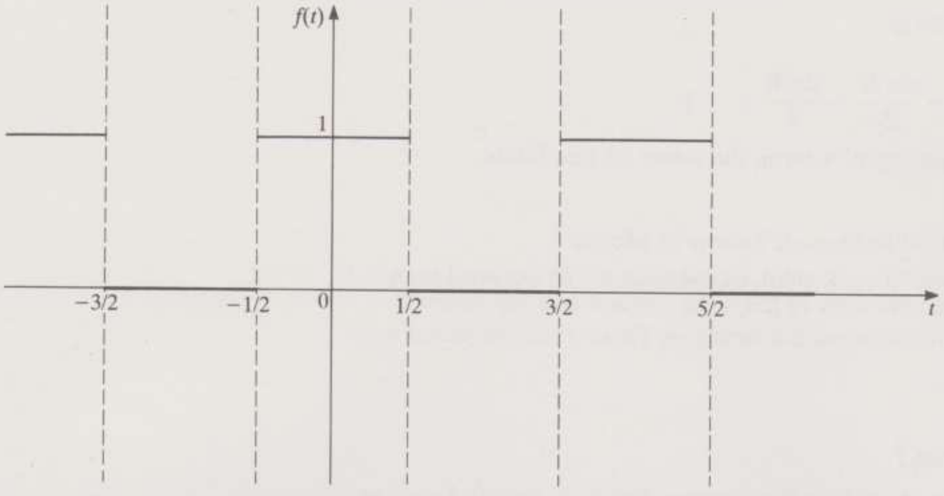


Figure 2. Square wave function. Note that the function is not defined at $t = \ldots, -3/2, -1/2, 1/2, 3/2, \ldots$.

[*Solution on p. 23*]

Exercise 4

Derive the Fourier series for the periodic function whose graph is shown in Figure 3.

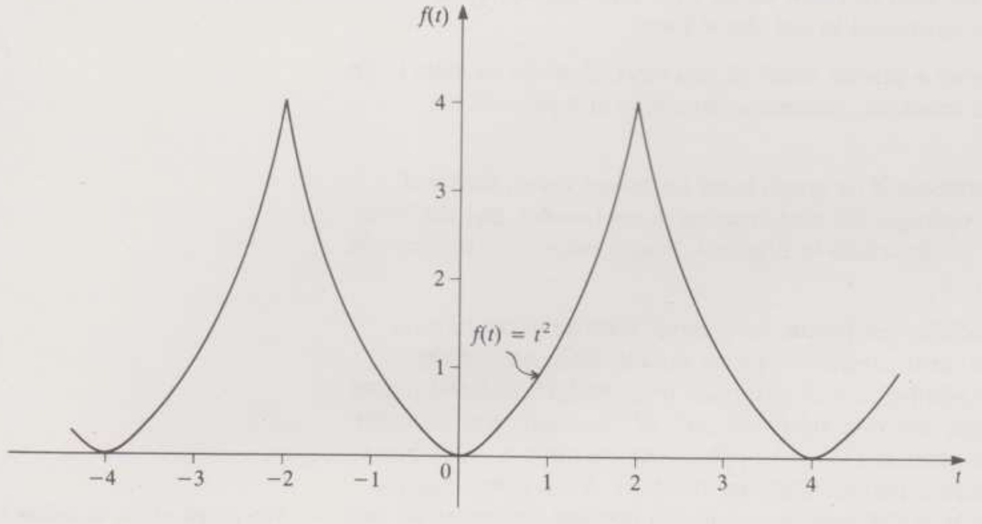


Figure 3

[*Solution on p. 24*]

Exercise 5

For the function shown in Figure 4, show that $M = B_n = 0$ and derive an expression for A_n.

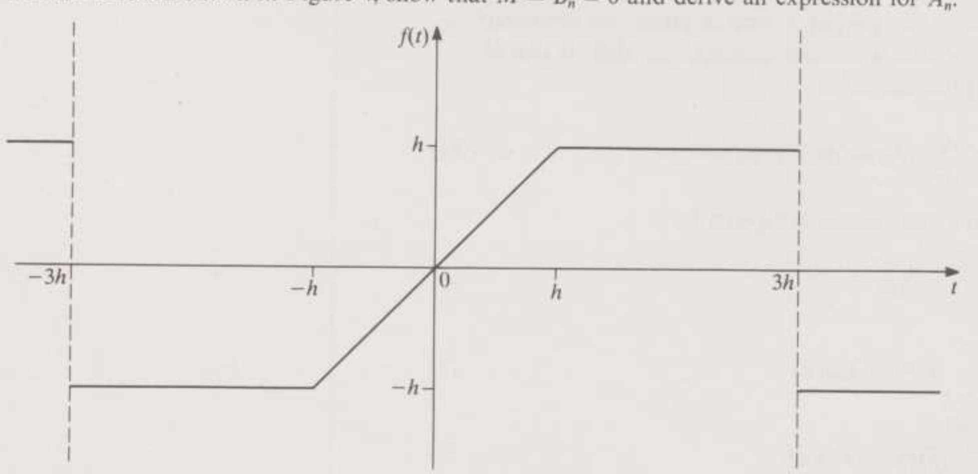


Figure 4

[*Solution on p. 24*]

Summary of Section 2

1. The **Fourier series** for a function $f(t)$ with period T is

$$f(t) = M + \sum_{n=1}^{\infty} (A_n \sin n\omega t + B_n \cos n\omega t),$$

where $\omega = 2\pi/T$.

2. The constants in the Fourier series are given by See Procedure 2.2.

$$M = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt \qquad (n = 1, 2, 3, \ldots)$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt \qquad (n = 1, 2, 3, \ldots).$$

3. (i) A function is said to be **continuous** if its graph is an unbroken curve.
(ii) A function is said to be **piecewise continuous** if its graph consists of continuous pieces and if there are a finite number of such pieces on any finite part of the domain. The end-points of the pieces must be well-defined.

4. Any periodic function which is piecewise continuous and whose derivative is also piecewise continuous can be represented by a Fourier series. Theorem 2.3.

3 Some outstanding points

3.1 Odd and even functions

The object of this subsection is to show you how to use the symmetry of the graph of $f(t)$ to save yourself work in calculating the Fourier series.

In Section 2 we derived the Fourier series (or at least the values of the coefficients) for the four functions which are sketched in Figure 1.

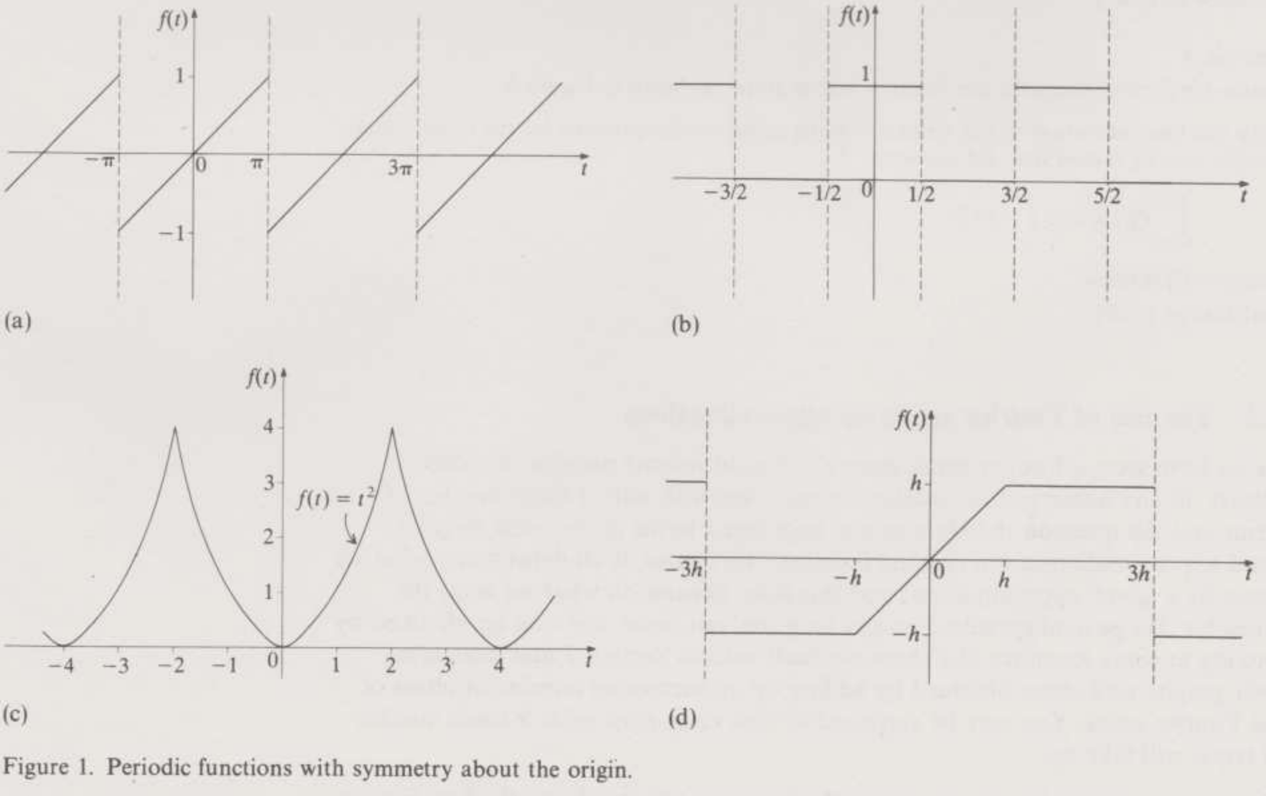


Figure 1. Periodic functions with symmetry about the origin.

All these four functions have some symmetry about the origin. But the symmetry takes two different forms: in Figures 1(b) and 1(c) the graph is symmetical under reflection in the vertical axis; this kind of symmetry is described by the formula

$f(t) = f(-t)$. This is the kind of symmetry which also applies to the function $\cos \omega t$, and a function with this symmetry is called an **even function**. On the other hand, in Figures 1(a) and 1(d) the graph is symmetrical under a rotation about the origin and this symmetry can be described by the formula $f(t) = -f(-t)$ which also applies to the function $\sin \omega t$. A function with this symmetry is called an **odd function**.

Now it is a fact that the Fourier series for an even function does not contain any sine terms (i.e. $A_n = 0$); apart from the constant term it can only consist of cosine terms. Similarly an odd function does not contain any cosine terms (i.e. $B_n = 0$) but only sine terms.

This piece of knowledge can save you the trouble of trying to evaluate coefficients which you can immediately predict will be zero. Look back at the four Fourier series in Section 2 and check that they conform with what I have said about odd and even functions.

Exercise 1

Figure 2 shows the graph of a periodic function of t in the shape of a triangular waveform. It is drawn so as to be symmetrical about the line $t = 0$. It is an odd or an even function?

[*Solution on p. 25*]

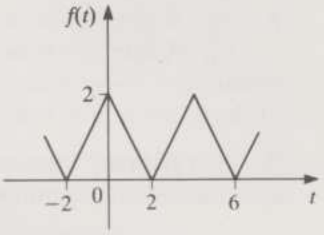


Figure 2. Triangular waveform.

If a function of t has no symmetry about the origin then it is neither odd nor even and its Fourier series will contain both sine and cosine terms.

Whether a function is odd or even or neither may depend merely on the choice of origin. For example, in Figure 2, if the graph were moved to the right by a distance less than a quarter of a period, the graph would no longer have symmetry about the origin and the Fourier series would contain both sine and cosine terms.

Exercise 2

State which of the quantities M, A_n, B_n will be zero for the function which results when, in Figure 2, the graph is moved

(i) one unit vertically downwards;
(ii) one unit vertically downwards and $\frac{1}{4}$ period to the left.

[*Solution on p. 25*]

Exercise 3

Derive the Fourier series for the function whose graph is shown in Figure 2.

(You can take advantage of the symmetry when doing the integrations for the terms which are not zero by noting that, for example,

$$\int_{-2}^{2} f(t)\,dt = 2\int_{0}^{2} f(t)\,dt$$

because $f(t)$ is even.)

[*Solution on p. 25*]

3.2 The use of Fourier series as approximations

As we have seen, a Fourier series generally has an infinite number of terms. Clearly, in any numerical calculation we can cope with only a finite number of terms and the question therefore arises: how many terms do we need to get a good approximation to the original function? Of course, it all depends on what we mean by a 'good' approximation; and this must depend on what we want the series for. No general specification can be given, but some 'feel' can be obtained by looking at some functions like those we dealt with in Section 2 and comparing their graphs with those obtained by adding up an increasing number of terms of the Fourier series. You may be surprised at how close even quite a small number of terms will take us.

By way of an example, we can start with the saw-tooth waveform, the first Fourier series we derived. Figure 3 shows the results of using from one to six terms of the series.

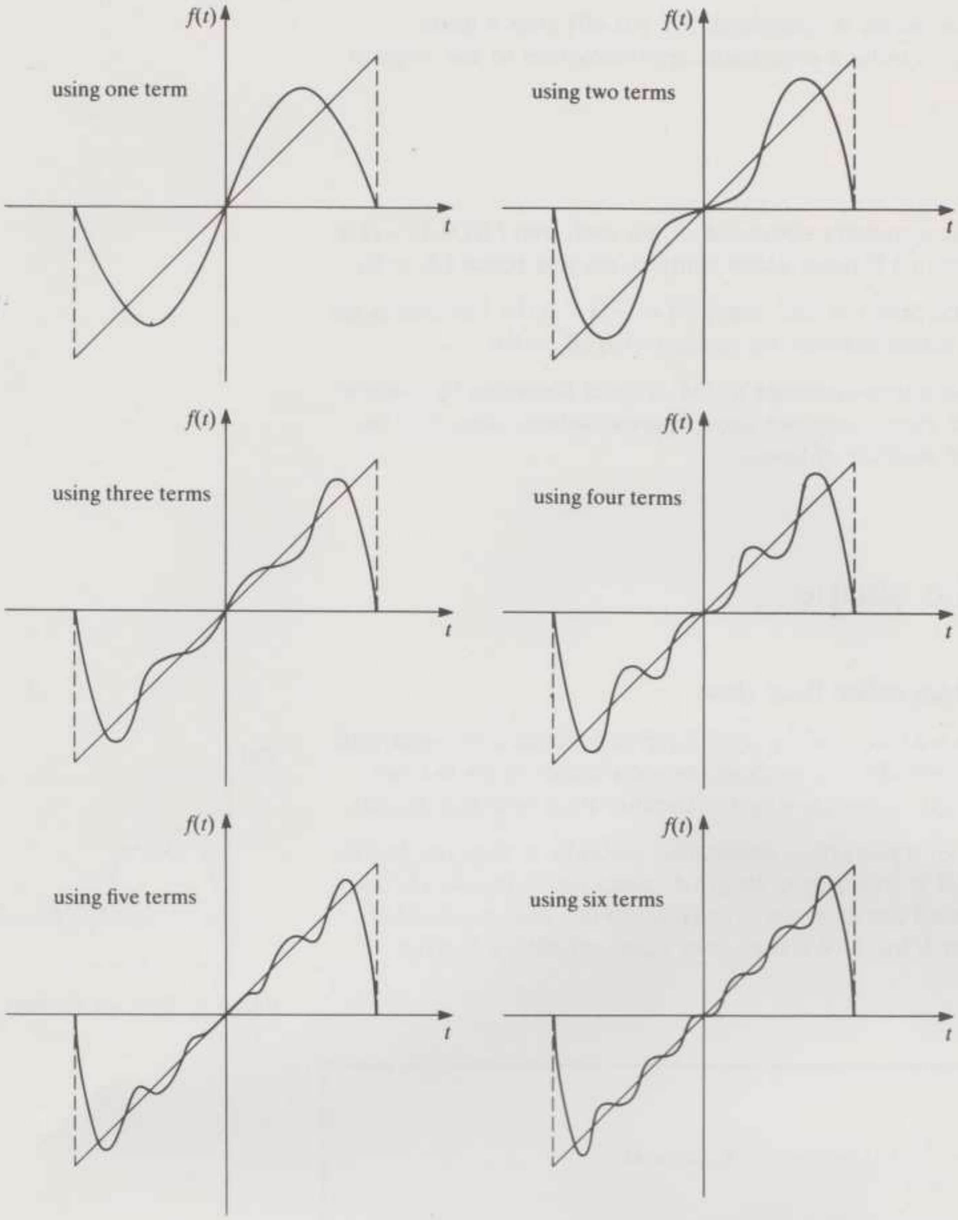


Figure 3. Graphs of partial sums of the terms of the Fourier series for the saw-tooth function.

Figure 4 shows a triangular wave with, superimposed, the fundamental component (curve I) and the sum of the first two components (curve II) of the Fourier series. Neither curve is very far removed from the original function.

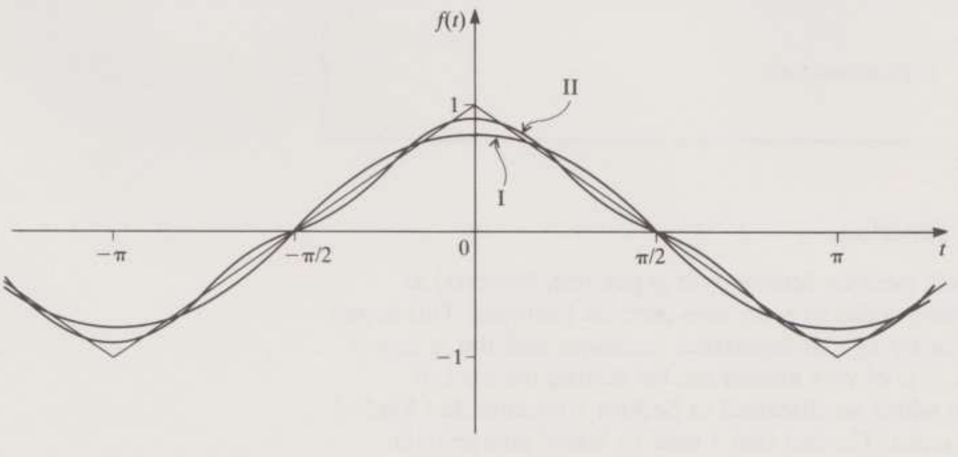


Figure 4. Triangular wave with the first two partial sums of the Fourier series $f(t) = \frac{8}{\pi^2}\,(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots)$.

Clearly, then, even if a Fourier series is 'truncated' (i.e. cut off) after a finite number of terms it can still give quite a reasonable approximation to the original function.

Summary of Section 3

1. A function which exhibits symmetry about the origin such that $f(t) = f(-t)$ is known as an **even function** and its Fourier series contains no sine terms ($A_n = 0$).

2. If the symmetry about the origin is such that $f(t) = -f(-t)$ the function is an **odd function** and its Fourier series contains no cosine terms ($B_n = 0$).

3. Fourier series are used as approximations to the original functions by using a limited number of terms. For many purposes good approximations can often be obtained with quite a modest number of terms.

4 Extending the scope

4.1 Functions of variables other than time

So far the main applications we have had in mind have been those concerned with variations of time. It is quite possible, by analogy, to use Fourier series in cases where the independent variable represents a quantity other than time (e.g. space).

Suppose we have a function of some other, unspecified variable, x. Suppose further that the function repeats itself at intervals of length L along the x-axis, as shown in Figure 1. We can derive the Fourier series exactly as we did before, provided that we substitute x for t and L for T. We shall then finish up with a Fourier series in x, as follows.

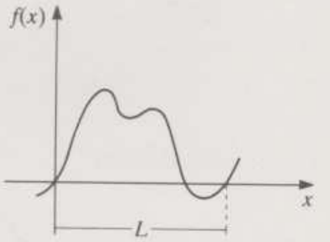


Figure 1. Periodic function of x.

Fourier series in x

$$f(x) = M + \sum_{n=1}^{\infty} (A_n \sin n\omega x + B_n \cos n\omega x)$$

where ω is given by $\frac{2\pi}{\omega} = L$, or $\omega = \frac{2\pi}{L}$

and $$M = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\,dx$$

$$A_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin n\omega x\,dx$$

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos n\omega x\,dx.$$

4.2 Some non-periodic functions

So far, we have dealt only with periodic functions. It is possible, however, to extend the techniques of Fourier series to some non-periodic functions. This aspect of the subject is important mainly as a mathematical technique and this is how it will be used in the next unit. It is of very limited use for solving the kind of 'frequency response' problem which we discussed in Section 1 because that kind of problem assumes a periodic input. The fact that I refer to 'some' non-periodic functions suggests that there is a restriction on the type of function we can cope with. The restriction is that the function must have a finite domain: it is defined over only a finite part of the t- or the x-axis.

The procedure for finding a Fourier series to represent such a function really amounts to extending the definition of the function to the whole of the t- or x-axis in such a way that the extended function is periodic. Then we can find the Fourier series for the extended function but use it only over the range for which the given function is defined. Let us take a simple example, the function $F(x)$ shown in Figure 2.

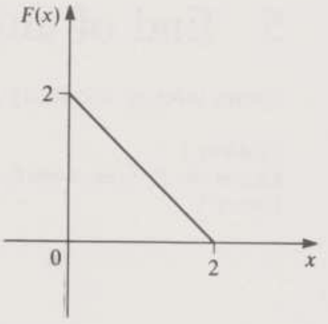


Figure 2

Here $F(x) = 2 - x$ and the domain consists of the interval of the x-axis between 0 and 2. The trick is to incorporate this function in a periodic function for which the Fourier series can easily be found. One way of doing this is shown in Figure 3.

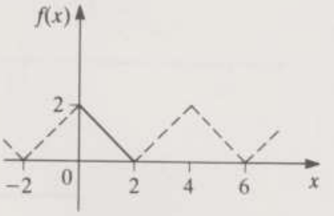


Figure 3

I have drawn dashed lines to define an even function $f(x)$ which is periodic with a period 4. Having got the Fourier series, we must remember to use it only over the range, $0 \le x \le 2$, of the actual non-periodic function we started with. The Fourier series is like the one we derived in Exercise 3 of Subsection 3.1. Substituting x for t, it is

$$f(x) = 1 + \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi x}{2} + \cdots\right) \qquad \text{(for all } x\text{)}.$$

This can be used to represent the function $F(x)$ shown in Figure 2 provided that we restrict ourselves to the domain $0 \le x \le 2$, i.e.

$$F(x) = 2 - x = 1 + \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi x}{2} + \cdots\right) \quad (0 \le x \le 2).$$

Exercise 1

Using the first four terms of the series derived above, investigate how accurately this partial sum represents

$$F(x) = 2 - x \qquad \text{for } 0 \le x \le 2$$

by substituting $x = 0$, $x = \frac{1}{2}$, $x = 1$, $x = 2$.

[*Solution on p. 25*]

Exercise 2

Again using only the first four terms, verify that the formula for $F(x)$ given above cannot be used to represent the function $2 - x$ when $x = 3$.

[*Solution on p. 25*]

Exercise 3

Sketch the graph of a periodic function which could be used to represent $F(x)$ as defined in Figure 2 by means of a series containing only sine terms. (Do not calculate the coefficients.)

[*Solution on p. 25*]

Exercise 4

Derive a Fourier cosine series with $M = \frac{1}{2}$ and period 4 to represent the non-periodic function shown in Figure 4, for the range $0 \le t < 1$.

[*Solution on p. 26*]

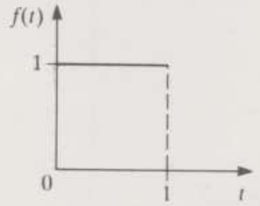


Figure 4

Summary of Section 4

1. The procedure for determining Fourier series for functions of time described in earlier sections can be used for functions of any variable x, with a period L, subject to substituting x for t and L for T.

2. A non-periodic function whose domain is a finite interval can be represented by a Fourier series by deriving the series for a periodic function of which the given function forms a part and then restricting the independent variable to the range of values of the variable for which the non-periodic function is defined.

5 End of unit problems

Please refer to the study guide for this unit before you begin work on this section.

Problem 1
Derive the Fourier series for the periodic function one complete cycle of which is shown in Figure 1.

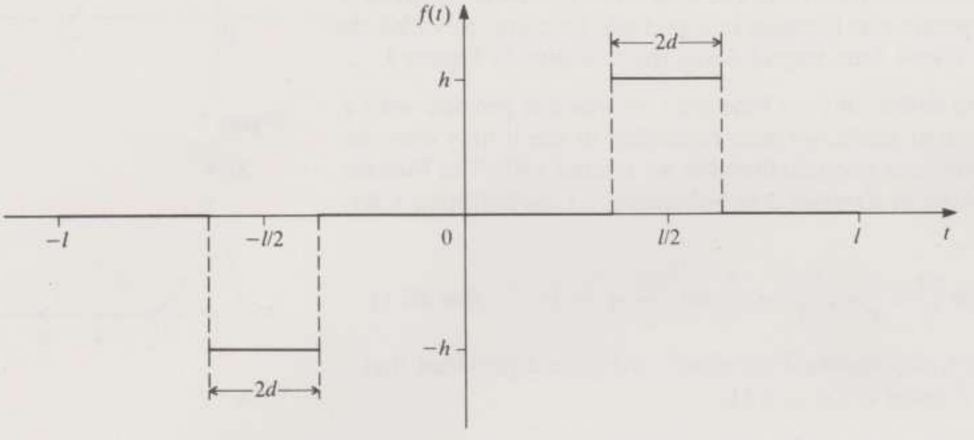


Figure 1

[*Solution on p. 26*]

Problem 2
Figure 2 represents the frequency response of an idealized device.

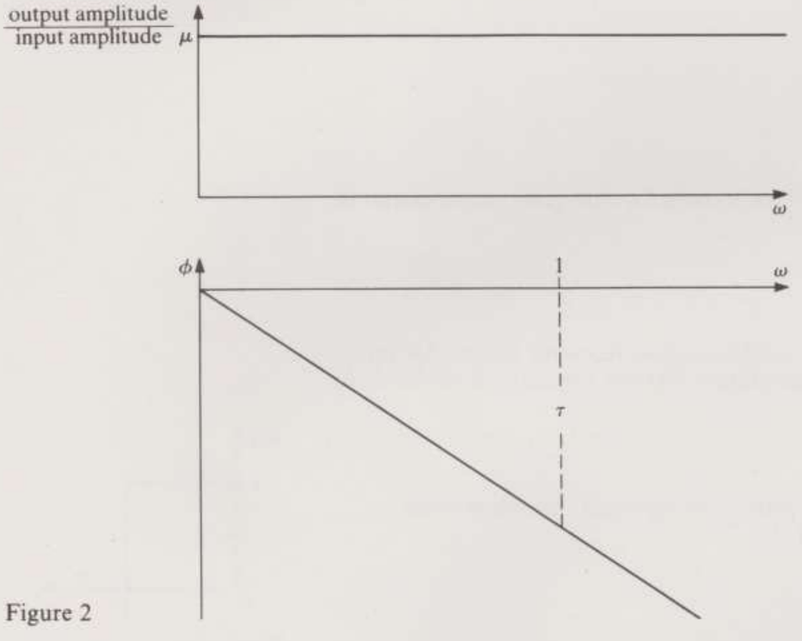


Figure 2

Show that, in response to any periodic input, the output of the device will be an exact scaled version of the input, shifted by an amount τ along the time axis.

[*Solution on p. 26*]

Problem 3
(i) Derive a Fourier sine series of period $2l$ to represent the non-periodic function whose graph is shown in Figure 3 over the range $0 \le x \le l$.

(ii) Assuming that we may neglect terms in the series having an amplitude equal to or less than 1% of the amplitude of the first term, how many terms of the Fourier series are required to represent $f(x)$ to this degree of approximation?

[*Solution on p. 27*]

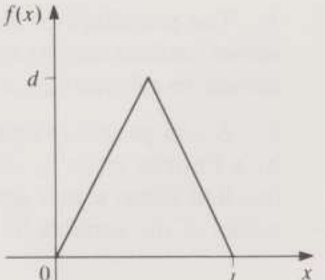


Figure 3

Appendix 1: Solutions to the exercises

Solutions to the exercises in Section 1

1. **(i)** The fundamental angular frequency is 2 radians per second.

(ii) The period is $\frac{2\pi}{2} = \pi$ seconds.

2. **(i)** Since the angular frequencies of all the components are integral multiples of the lowest one, i.e. $\omega = 4$, the period, T, of the function $f(t)$ will be $2\pi/4 = \pi/2$ seconds.

(ii) The first component has angular frequency $\omega = 4$. At this frequency the relative response, or amplitude ratio, is found from Figure 7 to be about 1.2, and the phase shift about $-11°$, or -0.19 radians. Hence the output corresponding to the term $\sin 4t$ is about $1.2\sin(4t - 0.19)$.

Similarly at $\omega = 8$, the amplitude ratio is about 2 and the phase shift about $-44°$ or -0.77 radians. The output component corresponding to the term $2\sin 8t$ is about $2 \times 2\sin(8t - 0.77) = 4\sin(8t - 0.77)$.

Again, at $\omega = 12$, the amplitude ratio is about 1.5 and the phase shift is about $-132°$ or -2.3 radians. The component of the output corresponding to the term $3\sin 12t$ is about $3 \times 1.5\sin(12t - 2.3)$, i.e. $4.5\sin(12t - 2.3)$.

Hence, by the superposition principle, the total response of the given linear system to the input $f(t)$ is

$$x(t) \simeq 1.2\sin(4t - 0.19) + 4\sin(8t - 0.77) + 4.5\sin(12t - 2.3).$$

(Your answers may differ slightly because the graph is hard to read accurately.)

Solutions to the exercises in Section 2

1.
$$\int_{-T/2}^{T/2} f(t)\cos\omega t\,dt = \int_{-T/2}^{T/2} M\cos\omega t\,dt + \int_{-T/2}^{T/2} A_1 \sin\omega t\cos\omega t\,dt + \int_{-T/2}^{T/2} B_1\cos^2\omega t\,dt + \cdots.$$

All the terms on the right-hand side except that containing B_1 are equal to zero. Hence

$$\int_{-T/2}^{T/2} f(t)\cos\omega t\,dt = \int_{-T/2}^{T/2} B_1\cos^2\omega t\,dt = B_1 \times \frac{T}{2}$$

Therefore

$$B_1 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\omega t\,dt.$$

2. **(i)** The period would still be 2π, and so $\omega = 1$ as before. Also, from the symmetry of the graph, $M = 0$ as before. So the new Fourier series has the form

$$f(t) = \sum_{n=1}^{\infty} (A_n \sin nt + B_n\cos nt).$$

Now, for $-\pi < t < \pi$, $f(t) = 2t/\pi$, so

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} \frac{2t}{\pi}\sin nt\,dt = \frac{2}{\pi^2}\int_{-\pi}^{\pi} t\sin nt\,dt.$$

This is just twice the original value so

$$A_n = \frac{4}{\pi n}(-1)^{n+1}.$$

Also

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} \frac{2t}{\pi}\cos nt\,dt$$

and again, since this is just twice the original value, $B_n = 0$.

So the new Fourier series is just twice the previous one:

$$f(t) = \frac{4}{\pi}\left(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \cdots\right).$$

(ii) The period is doubled, so $T = 4\pi$ and $\omega = \frac{1}{2}$. The Fourier series now has the form

$$f(t) = M + \sum_{n=1}^{\infty}\left(A_n \sin\frac{nt}{2} + B_n\cos\frac{nt}{2}\right).$$

For $-2\pi < t < 2\pi$, $f(t) = t/2\pi$, so

$$M = \frac{1}{4\pi}\int_{-2\pi}^{2\pi}\frac{t}{2\pi}\,dt = \frac{1}{8\pi^2}\left[\frac{t^2}{2}\right]_{-2\pi}^{2\pi} = 0$$

(as expected, since the mean value is still zero).

$$A_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}\frac{t}{2\pi}\sin\frac{nt}{2}\,dt = \frac{1}{4\pi^2}\int_{-2\pi}^{2\pi} t\sin\frac{nt}{2}\,dt$$

$$= \frac{1}{4\pi^2}\left[-\frac{2}{n}t\cos\frac{nt}{2} + \frac{4}{n^2}\sin\frac{nt}{2}\right]_{-2\pi}^{2\pi}$$

$$= \frac{2}{4\pi^2 n}[-2\pi\cos n\pi + 0 - 2\pi\cos(-n\pi) - 0]$$

$$= \frac{2}{\pi n}(-1)^{n+1}.$$

$$B_n = \frac{1}{4\pi^2}\int_{-2\pi}^{2\pi} t\cos\frac{nt}{2}\,dt$$

$$= \frac{1}{4\pi^2}\left[\frac{2}{n}t\sin\frac{nt}{2} + \frac{4}{n^2}\cos\frac{nt}{2}\right]_{-2\pi}^{2\pi} = 0.$$

So the new series is

$$f(t) = \frac{2}{\pi}\left(\sin(t/2) - \frac{\sin t}{2} + \frac{\sin(3t/2)}{3} - \cdots\right)$$

and the only difference is the changed value of ω.

3. From the graph, the period is 2. So $\omega = 2\pi/T = \pi$.

Here,

$$\int_{-T/2}^{T/2} f(t)\,dt = \int_{-1}^{1} f(t)\,dt = \int_{-1/2}^{1/2} 1\,dt$$

since over the rest of the cycle $f(t) = 0$. This time the mean value is clearly not equal to zero since no part of the waveform is below the axis. So

$$M = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{2}\int_{-1/2}^{1/2} 1\,dt = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}.$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt = 1\int_{-1/2}^{1/2} 1\sin n\pi t\,dt$$

$$= \left[-\frac{1}{n\pi}\cos n\pi t\right]_{-1/2}^{1/2}$$

$$= -\frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\left(\frac{-n\pi}{2}\right)\right] = 0.$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt = \int_{-1/2}^{1/2} \cos n\pi t\,dt$$
$$= \left[\frac{1}{n\pi}\sin n\pi t\right]_{-1/2}^{1/2} = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin\left(\frac{-n\pi}{2}\right)\right]$$
$$= \frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $B_1 = \frac{2}{\pi}\sin\frac{\pi}{2} \quad = \frac{2}{\pi}$

$$B_2 = \frac{2}{2\pi}\sin\pi \;= 0$$
$$B_3 = \frac{2}{3\pi}\sin\frac{3\pi}{2} = -\frac{2}{3\pi}$$
$$B_4 = \frac{2}{4\pi}\sin 2\pi = 0$$
$$B_5 = \frac{2}{5\pi}\sin\frac{5\pi}{2} = \frac{2}{5\pi} \qquad \text{and so on.}$$

Since the values of B_n are the coefficients of the cosine terms, the required Fourier series is:

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\pi t - \frac{\cos 3\pi t}{3} + \frac{\cos 5\pi t}{5} - \frac{\cos 7\pi t}{7} + \cdots\right).$$

The brackets contain an infinite series of cosine terms (the sine terms all being zero), and again, the higher the frequency the lower the amplitude.

4. From Figure 3, $T = 4 = \frac{2\pi}{\omega}$.

Hence $\omega = \frac{2\pi}{4} = \frac{\pi}{2}$.

The mean value $= M = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$

$$= \frac{1}{4}\int_{-2}^{2} t^2\,dt = \frac{1}{4}\left[\frac{t^3}{3}\right]_{-2}^{+2}$$
$$= \frac{1}{4}\left[\frac{8}{3} - \left(-\frac{8}{3}\right)\right] = \frac{1}{4}\times\frac{16}{3} = \frac{4}{3}.$$

$$A_n = \frac{2}{4}\int_{-2}^{+2} t^2 \sin n\omega t\,dt.$$

The integral tables in the *Handbook* give

$$\int t^2 \sin at\,dt = \left(-\frac{t^2}{a} + \frac{2}{a^3}\right)\cos at + \frac{2t}{a^2}\sin at + C.$$

Here, $a = n\omega$ so

$$A_n = \frac{1}{2}\left[\left(-\frac{t^2}{n\omega} + \frac{2}{n^3\omega^3}\right)\cos n\omega t\right]_{-2}^{+2}$$
$$+ \frac{1}{2}\left[\frac{2t}{n^2\omega^2}\sin n\omega t\right]_{-2}^{+2}$$
$$= \frac{1}{2n\omega}\left[\left(-4 + \frac{2}{n^2\omega^2}\right)\cos 2n\omega - \left(-4 + \frac{2}{n^2\omega^2}\right)\cos(-2n\omega)\right]$$
$$+ \frac{1}{n^2\omega^2}\left[2\sin 2n\omega - (-2\sin(-2n\omega))\right]$$
$$= 0.$$

Also $B_n = \frac{1}{2}\int_{-2}^{+2} t^2 \cos n\omega t\,dt.$

From the *Handbook*

$$\int t^2 \cos at\,dt = \left(\frac{t^2}{a} - \frac{2}{a^3}\right)\sin at + \frac{2t}{a^2}\cos at + C.$$

Now $a = n\omega$, as before, hence

$$B_n = \frac{1}{2}\left[\left(\frac{t^2}{n\omega} - \frac{2}{n^3\omega^3}\right)\sin n\omega t\right]_{-2}^{+2} + \frac{1}{2}\left[\frac{2t}{n^2\omega^2}\cos n\omega t\right]_{-2}^{+2}$$
$$= \frac{1}{2n\omega}\left[\left(4 - \frac{2}{n^2\omega^2}\right)\sin 2n\omega - \left(4 - \frac{2}{n^2\omega^2}\right)\sin(-2n\omega)\right]$$
$$+ \frac{1}{n^2\omega^2}\left[2\cos 2n\omega - (-2\cos(-2n\omega))\right].$$

But $\omega = \pi/2$, so that $\sin 2n\omega = \sin n\pi = 0$, and

$$B_n = \frac{4}{n^2\pi^2}\times 4\cos n\pi = \frac{16}{n^2\pi^2}(-1)^n.$$
$$B_1 = \frac{16}{\pi^2}\times(-1) = -\frac{16}{\pi^2}$$
$$B_2 = \frac{4}{\pi^2}\times(-1)^2 = \frac{4}{\pi^2}$$
$$B_3 = \frac{16}{9\pi^2}\times(-1)^3 = -\frac{16}{9\pi^2} \qquad \text{and so on.}$$

So

$$f(t) = \frac{4}{3} - \frac{16}{\pi^2}\left(\cos\frac{\pi}{2}t - \frac{1}{4}\cos\pi t + \frac{1}{9}\cos\frac{3\pi}{2}t + \cdots\right)$$
$$= \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{2}t.$$

5. From Figure 4 the period, $T = 6h$.

So $\omega = 2\pi/6h = \pi/3h$.

$$M = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{6h}\int_{-3h}^{3h} f(t)\,dt$$
$$= \frac{1}{6h}\left(\int_{-3h}^{-h} -h\,dt + \int_{-h}^{h} t\,dt + \int_{h}^{3h} h\,dt\right)$$
$$= \frac{1}{6h}(-2h^2 + 0 + 2h^2) = 0.$$

(In fact we can see directly from the graph that the mean value is zero.)

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt = \frac{1}{3h}\int_{-3h}^{3h} f(t)\sin n\omega t\,dt$$
$$= \frac{1}{3h}\left(\int_{-3h}^{-h} -h\sin n\omega t\,dt + \int_{-h}^{h} t\sin n\omega t\,dt + \int_{h}^{3h} h\sin n\omega t\,dt\right)$$

Using the integral tables in the *Handbook* to find the second integral

$$A_n = \frac{1}{3}\left[\frac{1}{n\omega}\cos n\omega t\right]_{-3h}^{-h}$$
$$+ \frac{1}{3h}\left[-\frac{t}{n\omega}\cos n\omega t + \frac{1}{n^2\omega^2}\sin n\omega t\right]_{-h}^{h}$$
$$+ \frac{1}{3}\left[-\frac{1}{n\omega}\cos n\omega t\right]_{h}^{3h}$$
$$= \frac{1}{3n\omega}\left[\cos(-n\omega h) - \cos(-3n\omega h) - \cos 3n\omega h + \cos n\omega h\right]$$
$$+ \frac{1}{3n\omega h}\left[-h\cos n\omega h + \frac{1}{n\omega}\sin n\omega h - h\cos(-n\omega h) - \frac{1}{n\omega}\sin(-n\omega h)\right].$$

The terms in cos $n\omega h$ cancel out, leaving

$$A_n = \frac{2}{3n\omega h}\left(\frac{1}{n\omega}\sin n\omega h - h\cos 3n\omega h\right).$$

But $\omega = \pi/3h$, so

$$A_n = \frac{2}{n\pi}\left(\frac{3h}{n\pi}\sin\frac{n\pi}{3} - h\cos n\pi\right).$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt = \frac{1}{3h}\int_{-3h}^{3h} f(t)\cos n\omega t\,dt$$

$$= \frac{1}{3h}\left(\int_{-3h}^{-h} - h\cos n\omega t\,dt + \int_{-h}^{h} t\cos n\omega t\,dt + \int_{h}^{3h} h\cos n\omega t\,dt\right).$$

Using the integral in the *Handbook* for the second integral,

$$B_n = \frac{1}{3}\left[-\frac{1}{n\omega}\sin n\omega t\right]_{-3h}^{-h} + \frac{1}{3h}\left[\frac{t}{n\omega}\sin n\omega t + \frac{1}{n^2\omega^2}\cos n\omega t\right]_{-h}^{h} + \frac{1}{3}\left[\frac{1}{n\omega}\sin n\omega t\right]_{h}^{3h}$$

$$= \frac{1}{3n\omega}\left[-\sin(-n\omega h) + \sin(-3n\omega h) + \sin 3n\omega h - \sin n\omega h\right]$$

$$+ \frac{1}{3n\omega h}\left[h\sin n\omega h + \frac{1}{n\omega}\cos n\omega h + h\sin(-n\omega h) - \frac{1}{n\omega}\cos(-n\omega h)\right]$$

$$= 0.$$

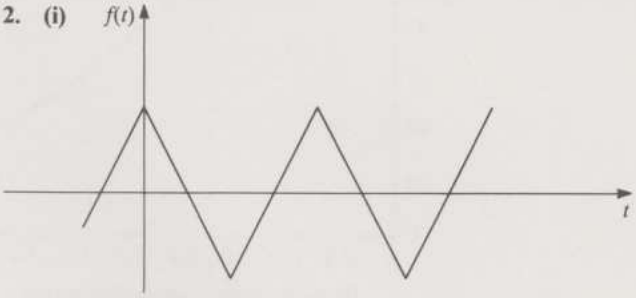
Hence $M = B_n = 0$ and

$$A_n = \frac{2h}{n\pi}\left(\frac{3}{n\pi}\sin\frac{n\pi}{3} - \cos n\pi\right).$$

Solutions to the exercises in Section 3

1. The function is even since $f(-t) = f(t)$, and its Fourier series would consist of a constant term (since its mean value is non-zero) and cosine terms.

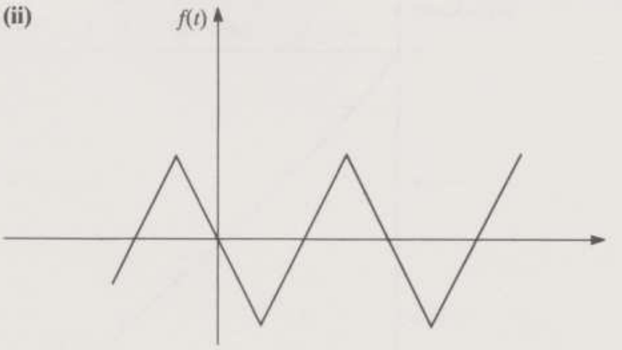
2. (i)



This will be an even function with zero mean value i.e.

$$M = A_n = 0, \qquad B_n \neq 0.$$

(ii)



This will be an odd function with zero mean value i.e.

$$M = B_n = 0, \qquad A_n \neq 0.$$

3. From the graph, $T = 4$, so $\omega = 2\pi/4 = \pi/2$.

$$M = \frac{1}{4}\int_{-2}^{2} f(t)\,dt = \frac{2}{4}\int_{0}^{2} f(t)\,dt.$$

Between 0 and 2, $f(t) = 2 - t$, so

$$M = \frac{1}{2}\int_0^2 (2-t)\,dt = \frac{1}{2}\left[2t - \frac{t^2}{2}\right]_0^2 = \frac{1}{2}(4-2) = 1.$$

Since the given function is even, $A_n = 0$.

$$B_n = \frac{2\times 2}{4}\int_0^2 (2-t)\cos\frac{n\pi t}{2}\,dt$$

$$= \int_0^2\left(2\cos\frac{n\pi t}{2} - t\cos\frac{n\pi t}{2}\right)dt$$

$$= \left[2\times\frac{2}{n\pi}\sin\frac{n\pi t}{2} - \frac{2t}{n\pi}\sin\frac{n\pi t}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right]_0^2$$

$$= \frac{4}{n\pi}\left(\sin n\pi - \sin n\pi - \frac{1}{n\pi}\cos n\pi - \sin 0 + 0 + \frac{1}{n\pi}\cos 0\right)$$

$$= \frac{4}{n^2\pi^2}(1 - \cos n\pi).$$

Thus $B_1 = \dfrac{8}{\pi^2}$, $B_2 = 0$, $B_3 = \dfrac{8}{9\pi^2}$, and so on.

Hence the required Fourier series is

$$f(t) = 1 + \frac{8}{\pi^2}\left(\cos\frac{\pi t}{2} + \frac{1}{9}\cos\frac{3\pi t}{2} + \frac{1}{25}\cos\frac{5\pi t}{2} + \cdots\right).$$

Solutions to the exercises in Section 4

1. $F(0) = 2$ and $1 + \dfrac{8}{\pi^2}\left(1 + \dfrac{1}{9} + \dfrac{1}{25}\right) \simeq 1.933.$

$F(\frac{1}{2}) = 1\frac{1}{2}$ and $1 + \dfrac{8}{\pi^2}(0.70711 - 0.07857 - 0.02828) \simeq 1.487.$

$F(1) = 1$ and $1 + \dfrac{8}{\pi^2}(0 + 0 + 0) = 1.$

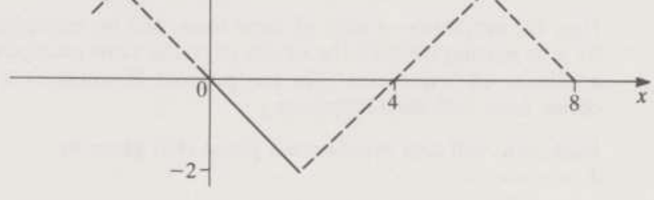
$F(2) = 0$ and $1 + \dfrac{8}{\pi^2}\left(-1 - \dfrac{1}{9} - \dfrac{1}{25}\right) \simeq 0.067.$

So that even with only a few terms the series gives an approximation to $F(x) = 2 - x$ with a maximum absolute error of 0.067 for the stated values of x.

2. $2 - 3 = -1$ and $1 + \dfrac{8}{\pi^2}(0 + 0 + 0) = 1.$

This confirms that outside the stated domain the series does not represent the original function.

3. The required graph is shown below.

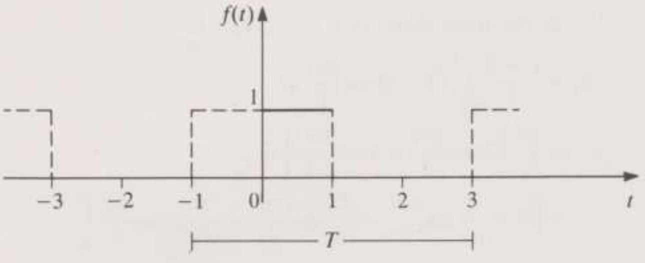


Note that the full line represents the function $G(x) = -x$, $0 \le x \le 2$, so that 2 would need to be added to the sine

series representing this graph in order to obtain the required representation of the function $F(x) = 2 - x$, $0 \le x \le 2$.

4. We need to incorporate the function shown in Figure 4 in a periodic function whose Fourier series will then represent the given function over its range of values of t, i.e. $0 \le t < 1$.

Exercise 3 in Section 2 provides a suitable function which, with a little adaptation, will give us what we want. We first draw the appropriate diagram in which only the given function appears in full lines.



This is like Figure 2 in Section 2, except that the period is different. Here $T = 4$, so that $\omega = \pi/2$ whereas in Section 2 $T = 2$ and $\omega = \pi$. This, however, is the only difference so that we can use the same series as we obtained then, changing only the frequencies of the terms. We therefore put

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t - \frac{1}{7}\cos\frac{7\pi}{2}t + \cdots\right) \quad \text{for } 0 \le t < 1.$$

You can check by substitution that this series works over the stated range $0 \le t < 1$. With a 'jump' discontinuity at $t = 1$, the series will give the average value of $f(t)$ for the vertical portion, i.e. for $t = 1$ the series gives $f(t) = \frac{1}{2}$.

Appendix 2: Solutions to the end of unit problems

1. From the graph in Figure 1, $T = 2l$, so $\omega = 2\pi/2l = \pi/l$. The graph is symmetric about the t-axis so $M = 0$. It is an odd function so the Fourier series has sine terms only, i.e. $B_n = 0$.

$$\begin{aligned}
A_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt \\
&= \frac{1}{l}\left(\int_{-\frac{l}{2}-d}^{-\frac{l}{2}+d} -h\sin\frac{n\pi t}{l}\,dt + \int_{\frac{l}{2}-d}^{\frac{l}{2}+d} h\sin\frac{n\pi t}{l}\,dt\right) \\
&= \frac{h}{l}\left\{\left[\frac{l}{n\pi}\cos\frac{n\pi t}{l}\right]_{-\frac{l}{2}-d}^{-\frac{l}{2}+d} + \left[-\frac{l}{n\pi}\cos\frac{n\pi t}{l}\right]_{\frac{l}{2}-d}^{\frac{l}{2}+d}\right\} \\
&= \frac{h}{n\pi}\left[\cos\left(\frac{n\pi}{l}\left(-\frac{l}{2}+d\right)\right) - \cos\left(\frac{n\pi}{l}\left(-\frac{l}{2}-d\right)\right) - \cos\left(\frac{n\pi}{l}\left(\frac{l}{2}+d\right)\right) + \cos\left(\frac{n\pi}{l}\left(\frac{l}{2}-d\right)\right)\right] \\
&= \frac{2h}{n\pi}\left[\cos n\pi\left(\frac{1}{2}-\frac{d}{l}\right) - \cos n\pi\left(\frac{1}{2}+\frac{d}{l}\right)\right].
\end{aligned}$$

Using the trigonometric identity from the *Handbook*

$$\sin\alpha\sin\beta = \tfrac{1}{2}\cos(\alpha-\beta) - \tfrac{1}{2}\cos(\alpha+\beta),$$

$$A_n = \frac{4h}{n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi d}{l}.$$

Hence

$$f(t) = \sum_{n=1}^{\infty}\frac{4h}{n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi d}{l}\sin\frac{n\pi t}{l}.$$

2. Let the input be $f(t)$ where

$$f(t) = M + A_1\sin\omega t + A_2\sin 2\omega t + A_3\sin 3\omega t + \cdots + B_1\cos\omega t + B_2\cos 2\omega t + B_3\cos 3\omega t + \cdots.$$

Now the amplitude of each of these terms will be multiplied by μ in passing through the device since the same multiplier applies to all frequencies. (For this purpose M counts as a cosine term with zero frequency.)

Each term will also experience a phase shift given by $\phi = -\tau\omega$.

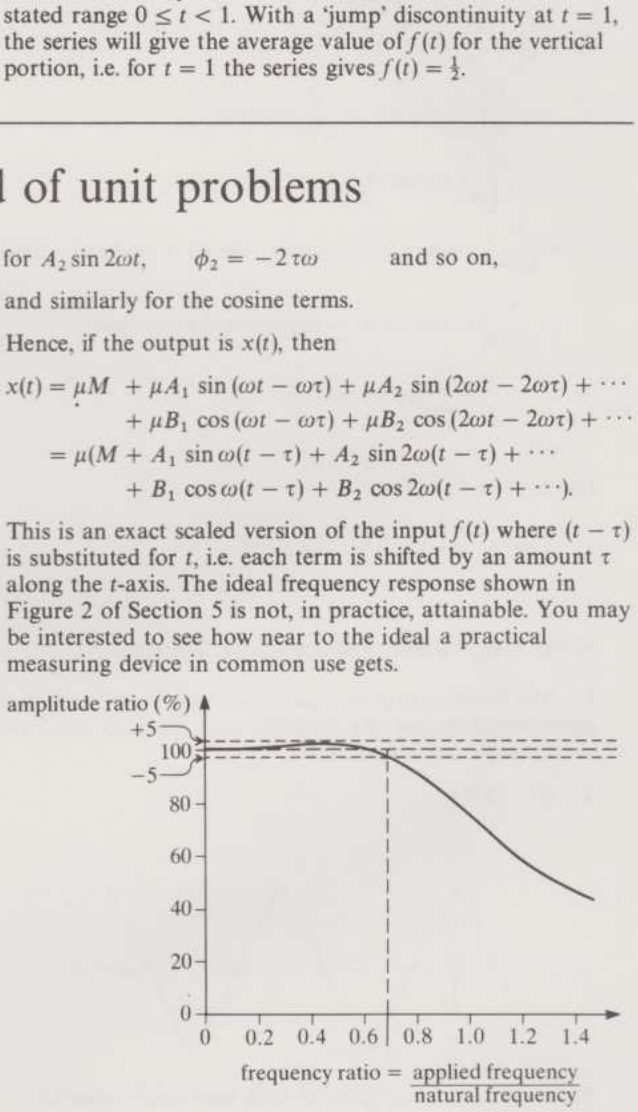
For M, $\quad \phi_M = \tau \times 0 = 0$,
for $A_1\sin\omega t$, $\quad \phi_1 = -\tau\omega$,
for $A_2\sin 2\omega t$, $\quad \phi_2 = -2\tau\omega \quad$ and so on,

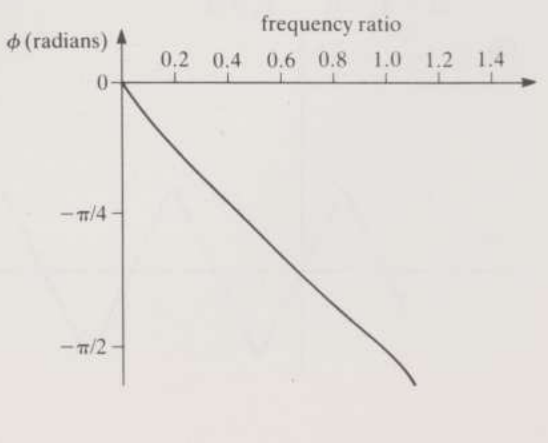
and similarly for the cosine terms.

Hence, if the output is $x(t)$, then

$$\begin{aligned}
x(t) &= \mu M + \mu A_1\sin(\omega t - \omega\tau) + \mu A_2\sin(2\omega t - 2\omega\tau) + \cdots \\
&\quad + \mu B_1\cos(\omega t - \omega\tau) + \mu B_2\cos(2\omega t - 2\omega\tau) + \cdots \\
&= \mu(M + A_1\sin\omega(t-\tau) + A_2\sin 2\omega(t-\tau) + \cdots \\
&\quad + B_1\cos\omega(t-\tau) + B_2\cos 2\omega(t-\tau) + \cdots).
\end{aligned}$$

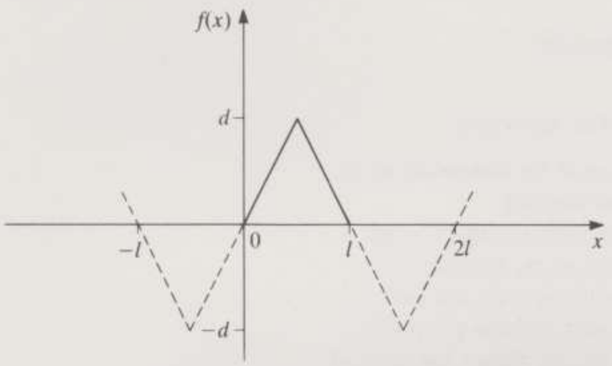
This is an exact scaled version of the input $f(t)$ where $(t - \tau)$ is substituted for t, i.e. each term is shifted by an amount τ along the t-axis. The ideal frequency response shown in Figure 2 of Section 5 is not, in practice, attainable. You may be interested to see how near to the ideal a practical measuring device in common use gets.





The above figure shows the frequency response graphs for a galvanometer on a commercially available voltage recording instrument. You can see that up to a frequency of about 0.7 times the natural frequency of the galvanometer, the actual relative response is within $\pm 5\%$ of the ideal, and the phase shift graph is quite close, too. For frequencies higher than about ($0.7 \times$ natural frequency) the relative response departs very markedly from the ideal. Hence, in order to get a true recording of a periodic input from this instrument, the terms of the Fourier series of the input with frequencies higher than ($0.7 \times$ natural frequency of galvanometer) must be negligible or zero.

3. **(i)** The question asks for a *sine* series, i.e. a series consisting only of sine terms; this means that we must find an odd function of which the given figure is a part. Such a function is shown below.



Here the 'fictitious' parts of the periodic function are drawn in dotted lines. For this periodic function, $M = B_n = 0$, and $L = 2l$, so that $\omega = 2\pi/L = \pi/l$.

$$
\begin{aligned}
A_n &= \frac{2}{2l}\int_{-l}^{l} f(x)\sin n\omega x\,dx \\
&= \frac{1}{l}\int_{-l}^{-l/2}\frac{2d}{l}(-x-l)\sin n\omega x\,dx + \frac{1}{l}\int_{-l/2}^{l/2}\frac{2d}{l}x\sin n\omega x\,dx \\
&\quad + \frac{1}{l}\int_{l/2}^{l}\frac{2d}{l}(-x+l)\sin n\omega x\,dx \\
&= \frac{2d}{l^2}\left[\frac{x}{n\omega}\cos n\omega x - \frac{1}{n^2\omega^2}\sin n\omega x + \frac{l}{n\omega}\cos n\omega x\right]_{-l}^{-l/2} \\
&\quad + \frac{2d}{l^2}\left[-\frac{x}{n\omega}\cos n\omega x + \frac{1}{n^2\omega^2}\sin n\omega x\right]_{-l/2}^{l/2} \\
&\quad + \frac{2d}{l^2}\left[\frac{x}{n\omega}\cos n\omega x - \frac{1}{n^2\omega^2}\sin n\omega x - \frac{l}{n\omega}\cos n\omega x\right]_{l/2}^{l} \\
&= \frac{2d}{l^2 n\omega}\left[-\frac{l}{2}\cos\left(-\frac{n\omega l}{2}\right) - \frac{1}{n\omega}\sin\left(-\frac{n\omega l}{2}\right) + l\cos\left(-\frac{n\omega l}{2}\right)\right. \\
&\quad + l\cos(-n\omega l) + \frac{1}{n\omega}\sin(-n\omega l) - l\cos(-n\omega l) \\
&\quad - \frac{l}{2}\cos\frac{n\omega l}{2} + \frac{1}{n\omega}\sin\frac{n\omega l}{2} - \frac{l}{2}\cos\left(-\frac{n\omega l}{2}\right) \\
&\quad - \frac{1}{n\omega}\sin\left(-\frac{n\omega t}{2}\right) + l\cos n\omega l - \frac{1}{n\omega}\sin n\omega l - l\cos n\omega l \\
&\quad \left. - \frac{l}{2}\cos\frac{n\omega l}{2} + \frac{1}{n\omega}\sin\frac{n\omega l}{2} + l\cos\frac{n\omega l}{2}\right] \\
&= \frac{2d}{l^2 n\omega}\left[\left(-\frac{l}{2} + l - \frac{l}{2} - \frac{l}{2} - \frac{l}{2} + l\right)\cos\frac{n\omega l}{2}\right. \\
&\quad + (l - l + l - l)\cos n\omega l \\
&\quad + \left(\frac{1}{n\omega} + \frac{1}{n\omega} + \frac{1}{n\omega} + \frac{1}{n\omega}\right)\sin\frac{n\omega l}{2} \\
&\quad \left. + \left(-\frac{1}{n\omega} - \frac{1}{n\omega}\right)\sin n\omega l\right] \\
&= \frac{2d}{l^2 n^2\omega^2}\left(4\sin\frac{n\omega l}{2} - 2\sin n\omega l\right).
\end{aligned}
$$

But $\omega = \dfrac{\pi}{l}$ so $\sin n\omega l = 0$ and

$$A_n = \frac{8d}{\pi^2 n^2}\sin\frac{n\pi}{2}.$$

Thus $A_1 = \dfrac{8d}{\pi^2}$, $A_2 = 0$, $A_3 = -\dfrac{8d}{9\pi^2}$, $A_4 = 0$, $A_5 = \dfrac{8d}{25\pi^2}$ and so on.

Hence the Fourier sine series is

$$f(x) = \frac{8d}{\pi^2}\left(\sin\frac{\pi x}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l} + \cdots\right), \quad \text{for } 0 \le x \le l$$

or

$$f(x) = \frac{8d}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l} \qquad \text{for } 0 \le x \le l.$$

(ii) The amplitude of the first term is $\dfrac{8d}{\pi^2}$.

The ratio of the amplitude of the nth term to that of the first term is $\dfrac{1}{(2n-1)^2}$.

For this to be greater than 1% we want $\dfrac{1}{(2n-1)^2} > \dfrac{1}{100}$ i.e. $(2n-1)^2 < 100$ or $2n - 1 < 10$. This means for $n \le 5$ we must include the terms in the Fourier series, so five terms are required.

Appendix 3: Television programme notes

You should read these notes before watching the television programme 'Fourier Analysis'.

A.1 Introduction

The television programme for this unit is about measurement and recording. It describes a procedure for predicting the response of a recording device to any periodic input that can be represented by a Fourier series. This procedure can be used for instruments such as audio amplifiers, record players or electrocardiographs. In the programme we use it to test a galvanometer which forms part of an ultra violet light recorder.

In order to use the procedure we need two pieces of information:

(i) the frequency response of the recording device;

(ii) the Fourier series of the periodic signal to be measured or recorded.

Since the Fourier series has an infinite number of terms it must be truncated to be used in practice, but enough terms must be left to constitute a good approximation to the original waveform. As I said in Subsection 3.2, there is no hard and fast definition of a 'good' approximation; the programme models the triangular wave, which serves as the main example, as the sum of only the first three terms of the relevant Fourier series. (You can get an idea of how good this approximation is by looking at Figure 4 in Subsection 3.2 which shows the sum of the first *two* terms in this Fourier series compared with the complete waveform.)

The programme also shows how the *relative response* of the galvanometer is found, by providing a constant-amplitude sinusoidal input over a range of frequencies and noting the variation in the output amplitude.

The relative response, i.e. $\dfrac{\text{output amplitude}}{\text{input amplitude}}$, is called the amplitude response in the television programme.

A.2 A note on frequency

In the programme, frequencies are quoted in cycles per second because instruments are calibrated in this way. However, in this unit we have worked exclusively in angular frequencies (radians per second). The connection between these quantities is quite simple.

If f = frequency, in cycles per second (or Hz)
and ω = angular frequency, in radians per second,
then

$$f = \frac{\omega}{2\pi}$$

and

$$\omega = 2\pi f.$$

The SI unit of frequency is the hertz:
$1\ \text{Hz} = 1\ \text{cycle}\ \text{s}^{-1}$.

f and ω are related to the period, T (in seconds), by
$f = \dfrac{1}{T}$, $\omega = \dfrac{2\pi}{T}$.

Example 1

(i) The fundamental frequency of a periodic waveform is 30 cycles per second. Calculate the corresponding value of the angular frequency.

(ii) The component with highest frequency in a truncated Fourier series has an angular frequency of 600 $\text{rad}\,\text{s}^{-1}$. What is its frequency in Hz?

Solution

(i) $\omega = 2\pi f \simeq 188.5$ (in radians per second).

(ii) $f = \dfrac{\omega}{2\pi} = \dfrac{600}{2\pi} \simeq 95.49$ (in Hz).

A.3 Relative response and phase shift

As you saw in Subsection 1.2, the frequency response of an instrument consists of its *relative response* (i.e. output amplitude/input amplitude) and its *phase shift*.

The relative response of the galvanometer in the ultra violet light recorder is shown in Figure 1(a). For good reproduction, the graph of the relative response should be as nearly as possible parallel to the horizontal axis over a range of frequencies. This range should include all the frequencies in the truncated Fourier series representing the signal to be recorded. In this case you can see that the graph is flat up to a frequency of about 90 cycles per second.

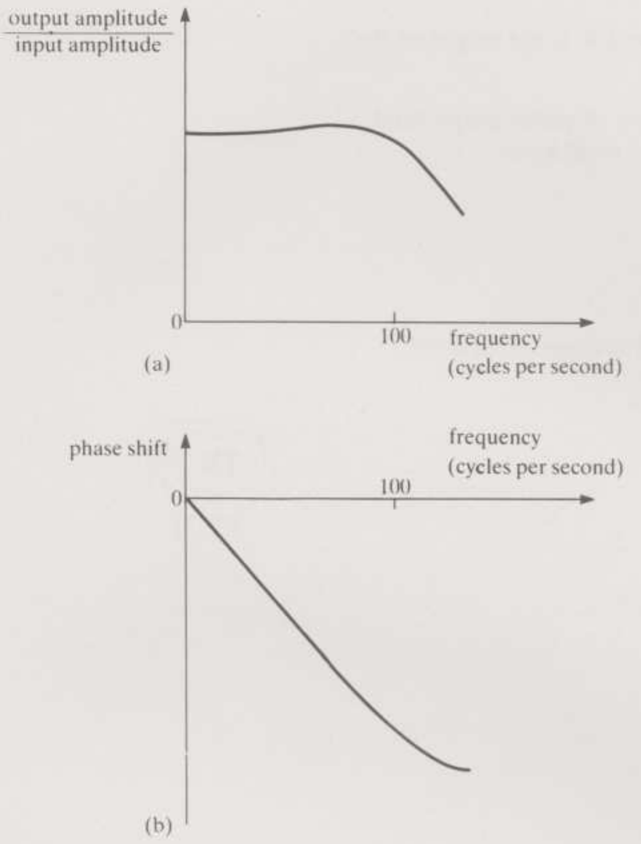


Figure 1. The frequency response of the galvanometer tested in the television programme.

The phase shift arises because the recording device takes time to respond to an input signal so that the output is not simultaneous with the input; each component of the Fourier series representing the input is delayed by an amount which depends on its frequency. Even if all the output amplitudes are correct, they will not add up to a good reproduction of the input if the instrument introduces a *time shift* of one component relative to another. In other words, the time shift should be the same for all components. To see what this means in terms of the frequency response, consider just one sinusoidal component, $f_i = F_i \sin \omega t$, say, of the input, $f(t)$. Then the corresponding component of the output, $x(t)$, will be $x_i = X_i \sin \omega(t - \tau)$, where τ is the delay introduced by the instrument. But in terms of the phase shift, ϕ, the output component is $x_i = X_i \sin(\omega t + \phi)$ so

$$\phi = -\omega\tau. \qquad (1)$$

We want the time shift, τ, to be constant for all values of ω, so Equation (1) shows that the graph of ϕ against ω should be a straight line. The relevant graph for the galvanometer is shown in Figure 1(b), and you can see that the frequency range over which this graph approximates a straight line is greater than the frequency range for which the relative response graph is flat. So provided we stay within the frequency limits imposed by the relative response, we shall automatically satisfy the phase shift requirements as well. For this reason the programme concentrates on the relative response.

Example 2

Assume that the triangular waveform shown in Figure 2 can be represented by the sum of the first three terms of its Fourier series; estimate the highest fundamental frequency at which such a wave will be reproduced by the galvanometer used in the programme.

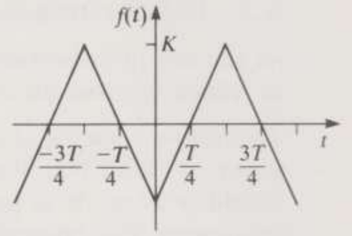


Figure 2

Solution

From the solution of Exercise 2(i) in Section 3, $M = A_n = 0$. Modifying the solution of Exercise 3 in Section 3, we find that the series is

$$f(t) = \frac{8K}{\pi^2}(\cos \omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t)$$

where $\omega = 2\pi/T$ is the fundamental angular frequency and K is the height of the triangular wave.

Now, in order to stay within the flat region of the relative response graph (and hence also the linear region of the phase shift graph), we must have

$$5\omega \leq 2\pi \times 90$$

i.e.

$$\omega \leq \frac{2\pi \times 90}{5} \simeq 113.$$

The fundamental frequency must be less than about 113 radians per second.

Now watch the television programme 'Fourier Analysis'.

TV

TV 31

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